

NEW TECHNIQUES FOR OBTAINING SCHUBERT-TYPE FORMULAS FOR HAMILTONIAN MANIFOLDS

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ABSTRACT. In [GT], Goldin and the second author extend some ideas from Schubert calculus to the more general setting of Hamiltonian torus actions on compact symplectic manifolds with isolated fixed points. (See also [Kn99] and [Kn08].) The main goal of this paper is to build on this work by finding more effective formulas.

More explicitly, given a generic component of the moment map, they define a *canonical class* α_p in the equivariant cohomology of the manifold M for each fixed point $p \in M$. When they exist, canonical classes form a natural basis of the equivariant cohomology of M . In particular, when M is a flag variety, these classes are the equivariant Schubert classes. It is a long standing problem in combinatorics to find positive integral formulas for the equivariant structure constants associated to this basis. Since computing the restriction of the canonical classes to the fixed points determines these structure constants, it is important to find effective formulas for these restrictions.

In this paper, we introduce new techniques for calculating the restrictions of a canonical class α_p to a fixed point q . Our formulas are nearly always simpler, in the sense that they count the contributions over fewer paths. Moreover, our formula is integral in certain important special cases.

INTRODUCTION

In [GT], Goldin and the second author extend some ideas from Schubert calculus to the more general setting of Hamiltonian torus actions on compact symplectic manifolds with isolated fixed points. (Knutson found closely related formulas for the Duistermaat-Heckman measure in the algebraic case in [Kn99] and [Kn08].) Given a generic component of the moment map, they define a canonical class α_p in the equivariant cohomology of the manifold M for each fixed point $p \in M$. When they exist, these canonical classes form a natural basis of the equivariant cohomology of M . In particular, when M is a flag variety, these classes are the equivariant Schubert classes (see [BGG]). It is a long standing problem in combinatorics to find positive integral formulas for the equivariant structure constants associated to this basis. Since computing the restriction of the canonical classes to the fixed points determines these structure constants and hence the (equivariant) cohomology ring of M , it is important to find effective formulas for these restrictions. Building on ideas of V. Guillemin and C. Zara [GZ], Goldin and Tolman show that the restriction of a canonical class α_p to a fixed point q can be calculated by a rational function which depends only on the following information: the value of the moment map at fixed points, and the restriction of other canonical classes to points of index exactly two higher. Moreover, the restriction formula in [GT] is *manifestly positive* whenever the restrictions themselves are all positive, including when M is a coadjoint orbit.

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However, the results in [GT] differ from Schubert calculus in several important ways. For example, that formula is almost never *integral*; essentially, this only holds when M is \mathbb{CP}^n . In contrast, in the combinatorics literature, a manifestly positive integral formula for the restriction of equivariant Schubert classes on $M = G/B$ is already known (see Appendix D.3 in [AJS] and [B]). The main goal of this paper is to bridge this gap by giving formulas which, like the formula in [GT], are valid in the much broader Hamiltonian category, but which are simpler in the sense that they count the contribution over fewer paths. Indeed, we want these contributions to be manifestly positive and integral whenever possible, and to understand geometrically when this occurs. This project was inspired by an early version of [Za], where C. Zara used combinatorial tools to rederive the integral formula in [AJS] and [B] for the case of a coadjoint orbit of type A_n from the formula in [GT], by taking limits as the cohomology class of the symplectic form varies.

In Theorem 2.1, we give a new formula for the restriction of a canonical class α_p to a fixed point q . This generalizes the formula in [GT] whenever $H^2(M; \mathbb{R}) \neq \mathbb{R}$. Like that formula, our formula is manifestly positive whenever the restrictions themselves are all positive. However, our formula is almost always simpler. For example, if M is a GKM space and J is an invariant compatible complex structure on M , then we can reduce the number of paths whenever there exists a cohomology class in the closure of the Kähler cone that vanishes on a sphere that is fixed by a codimension one subgroup, as long as the index of the two fixed points in the sphere differs by two; see Theorem 3.2.

Our technique is particularly powerful when the manifold is an equivariant fiber bundle over another Hamiltonian manifold so that, for example, the projection map intertwines compatible invariant complex structures; explicit computations are especially easy in this case. In particular, in Corollary 4.7 we give an inductive formula for the restrictions $\alpha_p(q)$ in terms of the paths in the base and the canonical classes on the fiber. Finally, by Theorem 4.4, our formula is *integral* whenever M is a “tower” of complex projective spaces, that is, a fiber bundle over \mathbb{CP}^n whose fiber is also a tower of complex projective spaces. More generally, if the fibers F_j are not projective spaces, but do satisfy $H^*(F_j; A) \simeq H^*(\mathbb{CP}^{n_j}; A)$ for some subring $A \subset \mathbb{R}$, then the contributions are all polynomials in the weights with coefficients in A .

Since coadjoint orbits of type A_n and C_n are both towers of complex projective spaces, we immediately get manifestly positive integral formulas for the restrictions in these cases. Similarly, since coadjoint orbits of type B_n are towers whose fibers satisfy $H^*(F_j; \mathbb{Z}[\frac{1}{2}]) \simeq H^*(\mathbb{CP}^{n_j}; \mathbb{Z}[\frac{1}{2}])$, the contribution of each path is integral when multiplied by a sufficiently large power of 2. (In a more recent version of [Za], Zara also independently was able to obtain formulas for C_n and B_n of this type as well.) Finally, coadjoint orbits of type B_n and D_n are sufficiently close to being towers of complex projective spaces that we can manipulate the terms to get manifestly positive integral formulas in these cases, as well.

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1. CANONICAL CLASSES

The main goal of this section is to define canonical classes and other important terminology, and to review the results that we will need in this paper. However, we also need to prove two slight variations of these results: Lemma 1.6 and Lemma 1.9.

Let T be a (compact) torus with Lie algebra \mathfrak{t} and lattice $\ell \subset \mathfrak{t}$, and let $\langle \cdot, \cdot \rangle$ be the natural pairing between \mathfrak{t}^* and \mathfrak{t} , where \mathfrak{t}^* is dual to \mathfrak{t} . Let T act on a compact symplectic manifold (M, ω) with moment map $\psi: M \rightarrow \mathfrak{t}^*$. By definition,

$$\iota_{X_\xi} \omega = -d\psi^\xi \quad \text{for all } \xi \in \mathfrak{t},$$

where X_ξ denotes the vector field on M generated by the action and $\psi^\xi(x) = \langle \psi(x), \xi \rangle$. In this case, we say that the triple (M, ω, ψ) is a **Hamiltonian T-manifold**.

Let $A \subset \mathbb{R}$ be a subring (with unit). The **equivariant cohomology of M with coefficients in A** is

$$H_T^*(M; A) = H^*(M \times_T ET; A);$$

it is naturally a module over $H^*(BT; A)$. Here, ET is a contractible space on which T acts freely, and $BT = ET/T$.

Now, assume that M has a discrete fixed set and fix a **generic** $\xi \in \mathfrak{t}$, that is, assume that $\langle \eta, \xi \rangle \neq 0$ for each weight $\eta \in \ell^* \subset \mathfrak{t}^*$ in the isotropy representation of T on $T_p M$ for every fixed point p . The function $\varphi = \psi^\xi: M \rightarrow \mathbb{R}$ is a Morse function; the critical set of φ is exactly the fixed set M^T . Hence, the index of φ at p is $2\lambda(p)$ for some $\lambda(p) \in \mathbb{N}$. In particular $H^1(M; \mathbb{R}) = 0$. For each $p \in M^T$, the negative normal bundle $\nu^-(p)$ of φ at p is a symplectic representation with no fixed sub-bundle, hence the individual weights of this representation are well-defined. Our convention for the moment map implies that these weights are exactly the **positive weights** of the isotropy action on $T_p M$, that is, the weights η such that $\langle \eta, \xi \rangle > 0$. Let $\Lambda_p^- \in \text{Sym}(\mathfrak{t}^*)$ be the product of these weights, where $\text{Sym}(\mathfrak{t}^*)$ denotes the symmetric algebra on \mathfrak{t}^* . Finally, given $\alpha \in H_T^*(M; A)$ and $q \in M^T$, let $\alpha(q)$ denote the image of α under the natural restriction map $H_T^*(M; A) \rightarrow H_T^*(\{q\}; A)$.

Definition 1.1. Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. A cohomology class $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ is a **canonical class** at a fixed point p (with respect to φ) if

- (1) $\alpha_p(p) = \Lambda_p^-$
- (2) $\alpha_p(q) = 0$ for all $q \in M^T \setminus \{p\}$ such that $\lambda(q) \leq \lambda(p)$.

Canonical classes do not always exist, but if they exist then they are unique [GT, Lemma 2.7]. Moreover, if there exist canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ for all $p \in M^T$, then by Lemmas 1.3 and 1.4 below, the classes $\{\alpha_p\}_{p \in M^T}$ are a basis for $H_T^*(M; A)$ as a module over $H^*(BT; A)$; see also [GT, Proposition 2.3]. In this case, our goal will be to compute the restrictions $\alpha_p(q)$ for all p and $q \in M^T$ in terms of paths in the canonical graph.

Definition 1.2. Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. There is a labelled directed graph (V, E) associated to $(M, \omega, \psi, \varphi)$, called the **canonical graph**, defined as follows.

- The vertex set V is the fixed set M^T ; we label each vertex $p \in V$ by its moment image $\psi(p) \in \mathfrak{t}^*$.

- The edge set is

$$E = \{(r, r') \in M^T \times M^T \mid \lambda(r') - \lambda(r) = 1 \text{ and } \alpha_r(r') \neq 0\};$$

we label each edge $(r, r') \in E$ by $\frac{\alpha_r(r')}{\Lambda_{r'}^-}$.

Given any directed graph with vertex set V and edge set $E \subset V \times V$, a **path** of **length** k from p to q is a $(k+1)$ -tuple $\gamma = (\gamma_1, \dots, \gamma_{k+1})$ so that $\gamma_1 = p$, $\gamma_{k+1} = q$, and $(\gamma_i, \gamma_{i+1}) \in E$ for all $1 \leq i \leq k$. For any path γ , we let $|\gamma|$ denote its length.

Throughout this paper, we will frequently need the following lemma, which is identical to [GT, Lemma 2.8] except that here we consider coefficients in any subring $A \subset \mathbb{R}$ instead of just \mathbb{Z} . The proof goes through without any change.

Lemma 1.3. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Given a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ at $p \in M^T$,*

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \setminus \{p\} \text{ such that } \varphi(q) \leq \varphi(p).$$

Lemma 1.3 implies that $\varphi(r) < \varphi(r')$ for all $(r, r') \in E$. Hence, if $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ is a path from p to q in (V, E) , then $\varphi(\gamma_i) < \varphi(q)$ for all $1 \leq i \leq |\gamma|$.

The following result is due to Kirwan, [Ki]; (see also [GT], [TW])

Lemma 1.4 (Kirwan). *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. For every fixed point p there exists a class $\gamma_p \in H_T^{2\lambda(p)}(M; \mathbb{Z})$ so that*

$$(1) \quad \gamma_p(p) = \Lambda_p^-, \text{ and}$$

$$(2') \quad \gamma_p(q) = 0 \text{ for every } q \in M^T \setminus \{p\} \text{ such that } \varphi(q) \leq \varphi(p).$$

Moreover, for any such classes, the $\{\gamma_p\}_{p \in M^T}$ are a basis for $H_T^*(M; \mathbb{Z})$ as a module over $H^*(BT; \mathbb{Z})$.

This has the following corollary, which we have adapted from [GT, Corollary 2.6] and [T, Corollary 2.3].

Corollary 1.5. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Fix $p \in M^T$ and $\beta \in H_T^{2i}(M; A)$ such that $\beta(q) = 0$ for all $q \in M^T$ so that $\varphi(q) < \varphi(p)$.*

- $\beta(p) = x \Lambda_p^-$ for some $x \in H^{2i-2\lambda(p)}(BT; A)$; in particular, if $\lambda(p) > i$ then $\beta(p) = 0$.
- Fix cohomology classes $\{\gamma_q\}_{q \in M^T}$ so that γ_q satisfies conditions (1) and (2') above for each $q \in M^T$. Then

$$\beta = \sum_{\varphi(q) \geq \varphi(p)} x_q \gamma_q, \quad \text{where } x_q \in H^{2i-2\lambda(q)}(BT; A) \text{ for all } q.$$

Here, the sum is over all $q \in M^T$ such that $\varphi(q) \geq \varphi(p)$.

We also need the following closely related fact.

Lemma 1.6. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes*

$\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. Fix $p \in M^T$ and $\beta \in H_T^{2i}(M; A)$ such that $\beta(q) = 0$ for all $q \in M^T$ so that $\lambda(q) < \lambda(p)$. Then

$$\beta = \sum_{\lambda(q) \geq \lambda(p)} x_q \alpha_q, \quad \text{where } x_q \in H^{2i-2\lambda(q)}(BT; A) \text{ for all } q.$$

Here the sum is over all $q \in M^T$ such that $\lambda(q) \geq \lambda(p)$.

Proof. Since $\{\alpha_q\}_{q \in M^T}$ is a basis for $H_T^*(M; A)$ as a module over $H^*(BT; A)$, we can write $\beta = \sum_{q \in M^T} x_q \alpha_q$, where $x_q \in H^{2i-2\lambda(q)}(BT; A)$ for all q . If the claim doesn't hold, then there exists $q \in M^T$ so that $\lambda(q) < \lambda(p)$ and $x_q \neq 0$, but $x_r = 0$ for all r such that $\lambda(r) < \lambda(q)$. Hence by the definition of canonical class $\beta(q) = x_q \Lambda_q^-$. Since $\beta(q) = 0$ this is impossible. \square

1.1. GKM spaces. We now restrict our attention to an important special case where it is especially easy to calculate canonical classes. A Hamiltonian T -manifold (M, ω, ψ) is a **GKM** (Goresky-Kottwitz-MacPherson) space if M has isolated fixed points and if, for every codimension one subgroup $K \subset T$, the fixed submanifold M^K has dimension at most two. Equivalently, M is a GKM space if the weights of the isotropy representation of T on $T_p M$ are pairwise linearly independent for every fixed point $p \in M^T$.

Definition 1.7. The **GKM graph** of a GKM space (M, ω, ψ) is the labelled directed graph (V, E_{GKM}) , defined as follows.

- The vertex set V is the fixed set M^T ; we label each $p \in M^T$ by its moment image $\psi(p) \in \mathfrak{t}^*$.
- Given $p \neq q$ in V , there is a directed edge $(p, q) \in E_{\text{GKM}}$ exactly if there exists a codimension one subgroup $K \subset T$ so that p and q are contained in the same connected component N of M^K . We label each edge (p, q) by the weight $\eta(p, q)$ associated to the isotropy representation of T on $T_q N \simeq \mathbb{C}$.

Observe that $(p, q) \in E_{\text{GKM}}$ exactly if $(q, p) \in E_{\text{GKM}}$. Moreover, $\eta(p, q) = -\eta(q, p)$, and $\psi(q) - \psi(p)$ is a positive multiple of $\eta(p, q)$ for all $(p, q) \in E_{\text{GKM}}$. Additionally, the set of weights of the isotropy representation on the tangent space at any point $p \in V$ is

$$\Pi_p = \Pi_p(M) = \{\eta(r, p) \mid (r, p) \in E_{\text{GKM}}\}.$$

Example 1.8 The complex projective space $\mathbb{C}P^n$ The natural action of $(S^1)^{n+1}$ on \mathbb{C}^{n+1} descends to an effective Hamiltonian action of $T = (S^1)^{n+1}/S^1$ on $\mathbb{C}P^n$. The associated GKM graph is the complete graph on $n+1$ fixed points: $p_1 = [1, 0, \dots, 0]$, $p_2 = [0, 1, \dots, 0], \dots$, $p_{n+1} = [0, 0, \dots, 1]$. Finally, the moment image of p_i is $\frac{1}{n+1} \sum_{j=1}^{n+1} (x_j - x_i)$, and the weight associated to the edge (p_i, p_j) is $x_i - x_j$. Here, we let x_1, \dots, x_{n+1} be the standard basis for $(\mathbb{R}^{n+1})^*$ and identify \mathfrak{t}^* with $\{\mu \in (\mathbb{R}^{n+1})^* \mid \sum \mu_i = 0\}$.

Now fix a generic component of the moment map $\varphi = \psi^\xi$. By the discussion at the beginning of this section, the set of weights in the isotropy representation on the negative normal bundle of φ at p is the set of positive weights in $\Pi_p(M)$. Hence, $\lambda(p)$ is the number of edges $(r, p) \in E_{\text{GKM}}$ such that $\varphi(r) < \varphi(p)$, and

$$\Lambda_p^- = \prod_{\substack{\eta \in \Pi_p(M) \\ \langle \eta, \xi \rangle > 0}} \eta.$$

It is possible to strengthen Lemma 1.3 when M is a GKM space. We say that a path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ in (V, E_{GKM}) is **ascending** if $\varphi(\gamma_i) < \varphi(\gamma_{i+1})$ for all i .

Lemma 1.9. *Let (M, ω, ψ) be a GKM space, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Given a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ at $p \in M^T$,*

$\alpha_p(q) = 0$ for all $q \in M^T$ if there are no ascending paths from p to q in (V, E_{GKM}) .

Proof. Consider any $q \in M^T$ so that $\alpha_p(q) \neq 0$ but $\alpha_p(r) = 0$ for each edge $(r, q) \in E_{\text{GKM}}$ such that $\varphi(r) < \varphi(q)$. Then $\alpha_p(q)$ is a non-zero multiple of $\eta(r, q)$ for all $(r, q) \in E_{\text{GKM}}$ such that $\varphi(r) < \varphi(q)$. (To see this, recall that for each $(r, q) \in E_{\text{GKM}}$ there exists a sphere $N \subset M$ containing r and q which is fixed by a codimension one subgroup $K \subset T$.) Since these weights are pairwise linearly independent, this implies that α_p has degree at least $2\lambda(q)$, that is, $\lambda(q) \leq \lambda(p)$. By the definition of canonical class, this is impossible unless $p = q$. The claim follows. \square

We say that φ is **index increasing** if $\lambda(p) < \lambda(q)$ for every edge $(p, q) \in E_{\text{GKM}}$ such that $\varphi(p) < \varphi(q)$. In this case, integral canonical classes exist and it is straightforward to compute the restriction of a canonical class α_p to q for any p and q in M^T such that $\lambda(q) - \lambda(p) = 1$. Conversely, if there exist canonical classes $\alpha_p \in H_T^*(M; \mathbb{Q})$ for all $p \in M^T$, then φ is index increasing [GT, Remark 4.2].

More specifically, let $\xi^\circ = \{\beta \in \mathfrak{t}^* \mid \langle \beta, \xi \rangle = 0\}$. Given $\eta \in \mathfrak{t}^*$, let $\rho_\eta: \text{Sym}(\mathfrak{t}^*) \rightarrow \text{Sym}(\mathfrak{t}^*)$ be the homomorphism of symmetric algebras induced by the projection map which sends $X \in \mathfrak{t}^*$ to $X - \frac{\langle X, \xi \rangle}{\langle \eta, \xi \rangle} \eta \in \xi^\circ \subset \mathfrak{t}^*$. Following [GZ], for any $(p, q) \in E_{\text{GKM}}$ we define

$$\Theta(p, q) = \frac{\rho_{\eta(p, q)}(\Lambda_p^-)}{\rho_{\eta(p, q)}\left(\frac{\Lambda_q^-}{\eta(p, q)}\right)} \in \text{Sym}(\mathfrak{t}^*)_0,$$

where $\text{Sym}(\mathfrak{t}^*)_0$ denotes the ring of fractions of $\text{Sym}(\mathfrak{t}^*)$. Observe that $\rho_{\eta(p, q)}\left(\frac{\Lambda_q^-}{\eta(p, q)}\right)$ is not zero, since by the GKM assumption the weights at each fixed point are pairwise linearly independent. The theorem below was proved in [GZ] over the rationals and then extended to the integers in [GT].

Theorem 1.10. *Let (M, ω, ψ) be a GKM space, and let (V, E_{GKM}) be the associated GKM graph. Let $\varphi = \psi^\xi$ be a generic component of the moment map; assume that φ is index increasing. Then*

- (1) *There exist canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; \mathbb{Z})$ for all $p \in M^T$.*
- (2) *Given fixed points p and q such that $\lambda(q) - \lambda(p) = 1$,*

$$\alpha_p(q) = \begin{cases} \Lambda_q^- \frac{\Theta(p, q)}{\eta(p, q)} & \text{if } (p, q) \in E_{\text{GKM}}, \text{ and} \\ 0 & \text{if } (p, q) \notin E_{\text{GKM}} \end{cases}$$

- (3) *$\Theta(p, q) \in \mathbb{Z} \setminus \{0\}$ for all $(p, q) \in E_{\text{GKM}}$ such that $\lambda(q) - \lambda(p) = 1$.*

In particular, the associated canonical graph has vertex set $V = M^T$ and edge set

$$E = \{(r, r') \in E_{\text{GKM}} \mid \lambda(r') - \lambda(r) = 1\}.$$

2. THE MOST GENERAL THEOREM

Our first main result is a generalization of [GT, Theorem 1.2]. More precisely, it is more general whenever $H^2(M; \mathbb{R}) \neq \mathbb{R}$. The main advantage of our formula is that in this case it nearly always allows us to express $\alpha_p(q)$ as a sum over fewer paths.

Theorem 2.1. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. Given fixed points p and q , let $\Sigma(p, q)$ denote the set of paths from p to q in the associated canonical graph (V, E) . Given classes $w_r \in H_T^2(M; \mathbb{R})$ for all $r \in M^T$,*

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in \Sigma(p, q)} \prod_{i=1}^{|\gamma|} \frac{w_{\gamma_i}(\gamma_{i+1}) - w_{\gamma_i}(\gamma_i)}{w_{\gamma_i}(q) - w_{\gamma_i}(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}$$

whenever the right hand side is well-defined, i.e., $w_{\gamma_i}(q) \neq w_{\gamma_i}(\gamma_i)$ for all $\gamma \in \Sigma(p, q)$ and $1 \leq i \leq |\gamma|$.

Remark 2.2. By Lemma 1.3, $\varphi(\gamma_i) < \varphi(q)$ for all $\gamma \in \Sigma(p, q)$ and $1 \leq i \leq |\gamma|$; a fortiori, $\psi(\gamma_i) \neq \psi(q)$. Therefore, the right hand side of the equation above is well-defined if w_r is a non-zero multiple of $[\omega + \psi]$ for all $r \in M^T$. (Here we are using the Cartan model for the equivariant cohomology of M .) In this case, the theorem agrees with [GT, Theorem 1.2].

Note that a path $\gamma \in \Sigma(p, q)$ contributes 0 to the formula above exactly if there exists $1 \leq i \leq |\gamma| - 1$ such that $w_{\gamma_i}(\gamma_i) = w_{\gamma_i}(\gamma_{i+1})$ but $w_{\gamma_i}(q) \neq w_{\gamma_i}(\gamma_i)$. Generally speaking, the best result will come from choosing each class w_r so that $w_r(r) \neq w_r(q)$, but $w_r(r) = w_r(s)$ for as many edges $(r, s) \in E$ as possible.

Proof of Theorem 2.1. Since $(w_p - w_p(p))(p) = 0$ and α_p is a canonical class at p , the restriction $\alpha_p(w_p - w_p(p))(r)$ is trivial for all $r \in M^T$ such that $\lambda(r) \leq \lambda(p)$. By Lemma 1.6, this implies that we can write

$$\alpha_p(w_p - w_p(p)) = \sum_{\lambda(r) > \lambda(p)} x_r \alpha_r, \quad \text{where } x_r \in H^{2\lambda(p) - 2\lambda(r) + 2}(BT; \mathbb{R}) \quad \text{for all } r.$$

By the definition of canonical class, evaluating the above equation at r implies that

$$(w_p(r) - w_p(p)) \frac{\alpha_p(r)}{\Lambda_r^-} = x_r \in \mathbb{R} \quad \text{for all } r \in M^T \text{ such that } \lambda(r) = \lambda(p) + 1.$$

Moreover, by dimensional arguments, $x_r = 0$ for all $r \in M^T$ such that $\lambda(r) > \lambda(p) + 1$. Hence,

$$\alpha_p(w_p - w_p(p)) = \sum_{(p, r) \in E} (w_p(r) - w_p(p)) \frac{\alpha_p(r)}{\Lambda_r^-} \alpha_r.$$

Restricting to q and dividing by $w_p(q) - w_p(p)$ (which is not zero by assumption), we have

$$\alpha_p(q) = \sum_{(p, r) \in E} \frac{w_p(r) - w_p(p)}{w_p(q) - w_p(p)} \frac{\alpha_p(r)}{\Lambda_r^-} \alpha_r(q).$$

Since the claim is obvious if $\lambda(q) - \lambda(p) \leq 1$, the claim now follows by induction. \square

The lemma below, which we will use in Section 4, follows from argument nearly identical to the first paragraph above.

Lemma 2.3. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$; let (V, E) be the canonical graph. Given a class $w \in H_T^2(M; A)$,*

$$(w(r) - w(p)) \frac{\alpha_p(r)}{\Lambda_r^-} \in A \quad \text{for all } (p, r) \in E.$$

In practice, instead of trying to pick the optimal class at each fixed point, we will often fix an order list of classes. For each fixed point we will just pick the first class that satisfies the hypotheses of Theorem 2.1. As we show below, as long as the forms satisfy the technical condition (1), this technique gives an elegant answer. In the next two sections, we will explain natural geometric conditions which guarantee that (1) is satisfied.

Corollary 2.4. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. Pick classes w_1, w_2, \dots, w_k in $H_T^2(M; \mathbb{R})$ such that, for each j ,*

$$(1) \quad \alpha_p(q) = 0 \quad \text{for all } p \text{ and } q \in M^T \text{ such that } w_j(q) \neq w_j(p) \text{ and } w_j^\xi(q) \leq w_j^\xi(p).$$

Assume that for each $(r, r') \in E$, there exists $j \in \{1, \dots, k\}$ such that $w_j(r) \neq w_j(r')$, and define

$$h(r, r') = \min \{j \mid w_j(r) \neq w_j(r')\} \quad \text{for all } (r, r') \in E.$$

Given p and q in M^T , let $\Sigma(p, q)$ denote the set of paths in the associated canonical graph (V, E) from p to q . Then

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in C(p, q)} \prod_{i=1}^{|\gamma|} \frac{w_{h(\gamma_i, \gamma_{i+1})}(\gamma_{i+1}) - w_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)}{w_{h(\gamma_i, \gamma_{i+1})}(q) - w_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}, \quad \text{where}$$

$$C(p, q) = \{ \gamma \in \Sigma(p, q) \mid h(\gamma_1, \gamma_2) \leq h(\gamma_2, \gamma_3) \leq \dots \leq h(\gamma_{|\gamma|}, \gamma_{|\gamma|+1}) \}.$$

Remark 2.5. Assume that for every positive k -tuple $a \in \mathbb{R}_+^k$ there exists a symplectic form $\omega_a \in \Omega^2(M)$ with moment map $\psi_a: M \rightarrow \mathfrak{t}^*$ such that

- (X) $[\omega_a + \psi_a] = \sum_i a_i w_i \in H_T^2(M; \mathbb{R})$, and
- (Y) the product of the positive weights for the isotropy representation of T on $(T_p M, \omega_a)$ is Λ_p^- for all $p \in M^T$.

By (Y), the canonical classes on M are the same for any symplectic form ω_a , where $a \in \mathbb{R}_+^k$. Therefore, in this case Corollary 2.4 can also be proven using the limit technique found in [Za].

In fact, if we assume that (X) holds for all $a \in \mathbb{R}_+^k$, then the w_i 's lie in the closure of one component of the Hamiltonian cone $\mathcal{H} \subset H_T^2(M; \mathbb{R})$. (See Definition 3.1.) If this is the component containing $[\omega + \psi]$, then Lemma 3.4 implies that (Y) automatically holds as well. Moreover, Lemma 3.5 shows that in this case (1) is satisfied, and so the conclusion of Corollary 2.4 holds. (See also Remark 2.2.) Thus, this is an important special case.

Proof. Assumption (1) implies that

$$\text{either } w_j^\xi(r) < w_j^\xi(r') \text{ or } w_j(r) = w_j(r') \text{ for all } (r, r') \in E \text{ and all } 1 \leq j \leq k.$$

In particular, if γ is a path from r to q with at least one edge, then $w_j^\xi(\gamma_2) \leq w_j^\xi(q)$. Since, by assumption, there exists $j \in \{1, \dots, k\}$ such that $w_j^\xi(\gamma_1) < w_j^\xi(\gamma_2)$, this implies that $w_j^\xi(r) < w_j^\xi(q)$. A fortiori, $w_j(r) \neq w_j(q)$, and so we can define

$$h(r, q) = \min \{j \mid w_j(r) \neq w_j(q)\} \quad \text{for all } r \in M^T \setminus \{q\} \text{ such that } \Sigma(r, q) \neq \emptyset.$$

Similarly, if $w_j(\gamma_i) = w_j(q)$ for some $\gamma \in \Sigma(p, q)$ and $i \leq |\gamma|$, then $w_j(\gamma_i) = w_j(\gamma_{i+1}) = w_j(q)$ as well. Therefore,

$$(2) \quad h(\gamma_i, q) \leq h(\gamma_{i+1}, q) \quad \text{and} \quad h(\gamma_i, q) \leq h(\gamma_i, \gamma_{i+1}) \quad \text{for all } 1 \leq i \leq |\gamma| - 1.$$

The assumptions of Theorem 2.1 will be satisfied if we let the class associated to $r \in M^T$ be

$$\begin{cases} w_{h(r, q)} & \text{if } r \neq q \text{ and } \Sigma(r, q) \neq \emptyset, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in \Sigma(p, q)} \prod_{i=1}^{|\gamma|} \frac{w_{h(\gamma_i, q)}(\gamma_{i+1}) - w_{h(\gamma_i, q)}(\gamma_i)}{w_{h(\gamma_i, q)}(q) - w_{h(\gamma_i, q)}(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}.$$

Moreover, a path $\gamma \in \Sigma(p, q)$ contributes 0 to the formula above if $w_{h(\gamma_i, q)}(\gamma_i) = w_{h(\gamma_i, q)}(\gamma_{i+1})$ for some $i < |\gamma|$. Therefore we only need to consider paths γ from p to q so that

$$h(\gamma_i, \gamma_{i+1}) \leq h(\gamma_i, q) \quad \text{for all } 1 \leq i \leq |\gamma|.$$

Combining this fact together with (2), we may restrict to paths γ so that

$$\begin{aligned} h(\gamma_1, \gamma_2) &\leq h(\gamma_2, \gamma_3) \leq \dots \leq h(\gamma_{|\gamma|}, \gamma_{|\gamma|+1}) \quad \text{and} \\ h(\gamma_i, \gamma_{i+1}) &= h(\gamma_i, q) \quad \text{for all } 1 \leq i \leq |\gamma|. \end{aligned}$$

□

3. THE HAMILTONIAN CONE

In this section, we give our first application of Theorem 2.1. Here, we use the Hamiltonian cone to pick the closed equivariant two-forms and characterize which paths contribute to the formula.

Definition 3.1. *Let a torus T act on a compact manifold M . The **Hamiltonian cone** of M , \mathcal{H} , is the set of classes in $H_T^2(M; \mathbb{R})$ which can be represented by an invariant symplectic form and moment map.*

Theorem 3.2. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. Pick classes w_1, \dots, w_k in the closure of the component of $\mathcal{H} \subset H_T^2(M; \mathbb{R})$ containing $[\omega + \psi]$. Assume that for each $(r, r') \in E$ there exists j such that $w_j(r) \neq w_j(r')$, and define*

$$h(r, r') = \min \{j \mid w_j(r) \neq w_j(r')\} \quad \text{for all } (r, r') \in E.$$

Given p and q in M^T , let $\Sigma(p, q)$ denote the set of paths from p to q in the associated canonical graph (V, E) . Then

$$\alpha_p(q) = \Lambda_q^- \sum_{\gamma \in C(p, q)} \prod_{i=1}^{|\gamma|} \frac{w_{h(\gamma_i, \gamma_{i+1})}(\gamma_{i+1}) - w_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)}{w_{h(\gamma_i, \gamma_{i+1})}(q) - w_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}, \quad \text{where}$$

$$C(p, q) = \{ \gamma \in \Sigma(p, q) \mid h(\gamma_1, \gamma_2) \leq h(\gamma_2, \gamma_3) \leq \cdots \leq h(\gamma_{|\gamma|}, \gamma_{|\gamma|+1}) \}.$$

Remark 3.3. Let $J: TM \rightarrow TM$ be a complex structure on M ; the **Kähler cone** of M is the set of classes in $H^2(M; \mathbb{R})$ which can be represented by a compatible symplectic form. If the complex structure is T -invariant then – by averaging – we can represent every such class by an invariant symplectic form. Hence if $H^1(M; \mathbb{R}) = 0$ the Kähler cone is a subset of the Hamiltonian cone. A analogous statement holds if J is an almost complex structure.

Note also that the Kähler cone is convex because any convex combination of compatible symplectic forms is itself a compatible symplectic form. In contrast, a convex combination of arbitrary symplectic forms may or may not be symplectic.

Lemma 3.4. Let (M, ω, ψ) be a Hamiltonian T -manifold, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Let ω' be a symplectic form on M with moment ψ' so that $[\omega' + \psi']$ lies in the component of $\mathcal{H} \subset H_T^2(M; \mathbb{R})$ containing $[\omega + \psi]$. Then $(\Lambda_p')^-$, the product of the positive weights of the isotropy representation of T on $(T_p M, \omega')$, is Λ_p^- for all $p \in M^T$.

Proof. Let $\varphi' = (\psi')^\xi$. It is sufficient to prove the claim for all ω' such that $[\omega']$ lies in some neighborhood of $[\omega]$. Therefore, we may assume that

$$\varphi(r) < \varphi(s) \Rightarrow \varphi'(r) < \varphi'(s) \quad \text{for all } r \text{ and } s \in M^T.$$

Fix $p \in M^T$. By applying Lemma 1.4 to φ , there exists a class $\gamma_p \in H_T^{2\lambda(p)}(M; \mathbb{Z})$ so that $\gamma_p(p) = \Lambda_p^-$ and $\gamma_p(q) = 0$ for every $q \in M^T \setminus \{p\}$ such that $\varphi(q) \leq \varphi(p)$. By the assumption above, this implies that $\gamma_p(q) = 0$ for every $q \in M^T \setminus \{p\}$ such that $\varphi'(q) < \varphi'(p)$. By applying Corollary 1.5 to φ' , this implies that $\Lambda_p^- = \gamma_p(p)$ is a multiple of $(\Lambda_p')^-$. Since a nearly identical argument shows that $(\Lambda_p')^-$ is a multiple of Λ_p^- , the claim follows from the fact that these are both products of positive weights. \square

Lemma 3.5. Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Fix a class w in the closure of the component of $\mathcal{H} \subset H_T^2(M; \mathbb{R})$ containing $[\omega + \psi]$. Given a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ at $p \in M^T$ (with respect to φ),

$$\alpha_p(q) = 0 \quad \text{for all } p \text{ and } q \in M^T \text{ such that } w(q) \neq w(p) \text{ and } w^\xi(q) \leq w^\xi(p).$$

Proof. By perturbing ξ slightly, if necessary, we may assume that $w(p) = w(q)$ exactly if $w^\xi(p) = w^\xi(q)$ for all p and q in M^T . Hence, there exists $\epsilon > 0$ so that $w^\xi(q) < w^\xi(p) - \epsilon$ for all $q \in M^T$ such that $w(q) \neq w(p)$ and $w^\xi(q) \leq w^\xi(p)$. By assumption, there exists a symplectic form ω' with moment map ψ' so that

$$(3) \quad |(\psi')(p) - w(p)| < \epsilon \quad \text{for all } p \in M^T,$$

and $[\omega' + \psi']$ lies in the closure of the component of $\mathcal{H} \subset H_T^2(M; \mathbb{R})$ containing $[\omega + \psi]$.

By Lemma 3.4, the product of the positive weights for the isotropy action on $(T_p M, \omega')$ is Λ_p^- for all $p \in M^T$. Hence, by the definition of canonical class, α_p is also the canonical class at p with respect to φ' . By Lemma 1.3, this implies that

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \text{ such that } (\psi')^\xi(q) < (\psi')^\xi(p).$$

Finally, (3) implies that $(\psi')^\xi(q) < (\psi')^\xi(p)$ for all $q \in M^T$ such that $w^\xi(q) < w^\xi(p) - \epsilon$. \square

Proof of Theorem 3.2. The claim follows immediately from Lemma 3.5 and Corollary 2.4. \square

Finally, we make the following observation, which we will not need in this paper.

Lemma 3.6. *Let (M, ω, ψ) be a Hamiltonian T -manifold. Let ω' be a symplectic form on M with moment ψ' so that $[\omega' + \psi']$ lies in the component of $\mathcal{H} \subset H_T^2(M; \mathbb{R})$ containing $[\omega + \psi]$. Then $c_1(M) = c'_1(M)$, where $c_1(M)$ and $c'_1(M) \in H_T^2(M; \mathbb{Z})$ are the first equivariant Chern class associated to ω and ω' , respectively.*

Proof. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Since the weights in the representations $(T_q M, \omega)$ and $(T_q M, \omega')$ agree up to sign, Lemma 3.4 implies immediately that $c_1(M)(q) = c'_1(M)(q)$ for all $q \in M^T$ such that $\lambda(q) \leq 1$. By Lemma 1.6, this implies that $c_1(M) - c'_1(M) = 0$. \square

4. FIBER BUNDLES

In this section, we show how to use Theorem 2.1 (and Corollary 2.4) to get effective formulas for the restrictions $\alpha_p(q)$ in the case that our Hamiltonian T -manifold is a fiber bundle over a Hamiltonian T -manifold (and certain technical restrictions hold). In certain very nice cases, such as when M is a “tower” of complex projective spaces and the restrictions of the canonical classes are positive, the contribution from each path will be a positive integer multiple of the product of (certain) positive weights. More precisely, let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds. We will consider the following maps.

Definition 4.1. *A map $\pi: M \rightarrow \widetilde{M}$ is a **strong symplectic fibration**¹ if*

- (1) *the map π is an equivariant fiber bundle with **symplectic fibers**, that is, the restriction of ω to the fiber $\widehat{M}_p = \pi^{-1}(\pi(p))$ is symplectic for all $p \in M$; and*
- (2) *as symplectic representations $(T_p M, \omega) \simeq (T_p \widehat{M}_p, \omega|_{\widehat{M}_p}) \oplus (T_{\pi(p)} \widetilde{M}, \widetilde{\omega})$ for all $p \in M^T$.*

Example 4.2 There are several situations where an equivariant fiber bundle $\pi: M \rightarrow \widetilde{M}$ is *automatically* a strong symplectic fibration.

- (i) Let J and \widetilde{J} be compatible almost complex structures on M and \widetilde{M} , respectively. If $\pi: M \rightarrow \widetilde{M}$ intertwines J and \widetilde{J} , i.e. $d\pi \circ J = \widetilde{J} \circ d\pi$, then π is a strong symplectic fibration. The fibers are symplectic because $T_p \widehat{M}_p$ is J invariant for all $p \in M$. For all $p \in M^T$, the symplectic perpendicular $H_p = (T_p \widehat{M}_p)^\omega$ is a complex subspace and $T_p M = T_p \widehat{M}_p \oplus H_p$ as complex representations. Finally, π induces an isomorphism of $(H_p, \omega|_{H_p})$ and $(T_{\pi(p)} \widetilde{M}, \widetilde{\omega})$ as symplectic representations.

¹Every map satisfying (1) is a symplectic fibration; see [MS, Lemma 6.2].

- (ii) If π has symplectic fibers and $\omega_\mu = \mu\omega + (1-\mu)\pi^*(\tilde{\omega})$ is symplectic for all $\mu \in (0, 1]$, then π is a strong symplectic fibration. Since $(0, 1]$ is connected, $(T_p M, \omega_\mu) \simeq (T_p M, \omega)$ as symplectic representations for all $\mu \in (0, 1]$ and all $p \in M^T$. But by Lemma 4.13, for any sufficiently small $\mu > 0$, $(T_p M, \omega_\mu) \simeq (T_p \widehat{M}_p, \omega|_{\widehat{M}_p}) \oplus (T_{\pi(p)} \widetilde{M}, \tilde{\omega})$ for all $p \in M^T$.

Remark 4.3. Let $\pi: M \rightarrow \widetilde{M}$ be any equivariant fiber bundle with symplectic fibers. Then $(T_p M, \omega) \simeq (T_p \widehat{M}_p, \omega|_{\widehat{M}_p}) \oplus (H_p, \omega|_{H_p})$ for all $p \in M^T$, where $H_p \subset M_p$ is the symplectic perpendicular to $T_p \widehat{M}_p$. Moreover, $d\pi: H_p \rightarrow T_{\pi(p)} \widetilde{M}$ is an isomorphism, and so the weights in the symplectic representations $(H_p, \omega|_{H_p})$ and $(T_{\pi(p)} \widetilde{M}, \tilde{\omega})$ necessarily agree up to sign. The map π is a strong symplectic fibration if they agree exactly.

Notation: Given a strong symplectic fibration $\pi: M \rightarrow \widetilde{M}$ and a generic component of the moment map $\varphi = \psi^\xi: M \rightarrow \mathbb{R}$, let $\widehat{\Lambda}_p^-$ denote the equivariant Euler class of the negative normal bundle of the restriction $\varphi|_{\widehat{M}_p}$ at $p \in \widehat{M}_p$, and let $2\widehat{\lambda}(p)$ denote the index of p in \widehat{M}_p , for all $p \in M^T$. (Since the restriction of ω to \widehat{M}_p is symplectic, the restriction of ψ to \widehat{M}_p is a moment map.) Similarly, let $\widetilde{\Lambda}_q^-$ denote the equivariant Euler class of the negative normal bundle of $\widetilde{\varphi} = \widetilde{\psi}^\xi$ at q and let $2\widetilde{\lambda}(q)$ denote the index of $q \in \widetilde{M}$, for all $q \in \widetilde{M}^T$.

Finally, given a subring $A \subseteq \mathbb{R}$, let $A_+ = \{t \in A \mid t > 0\}$ and let $A^\times \subset A$ denote the set of units.

We can now state our main theorem in this section.

Theorem 4.4. Let $\{(M_j, \omega_j, \psi_j)\}_{j=0}^k$ be Hamiltonian T -manifolds with discrete fixed sets so that M_0 is a point, and let $\{p_j: M_{j+1} \rightarrow M_j\}_{j=0}^{k-1}$ be strong symplectic fibrations. Let $\varphi_k = \psi_k^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\widehat{\lambda}(p)}(M_k; A)$ exist for all $p \in M_k^T$. Let² $\pi_j = p_j \circ p_{j+1} \circ \cdots \circ p_{n-1}: M_k \rightarrow M_j$ and let $\overline{\psi}_j = \pi_j^*(\psi_j): M_k \rightarrow \mathfrak{t}^*$ for all j . Finally, define

$$h(r, r') = \min\{j \in \{1, \dots, k\} \mid \pi_j(r) \neq \pi_j(r')\} \quad \text{for all distinct } r \text{ and } r' \text{ in } M^T.$$

1. Given p and q in M_k^T , let $\Sigma(p, q)$ denote the set of paths from p to q in the associated canonical graph (V, E) ; then

$$\alpha_p(q) = \sum_{\gamma \in C(p, q)} \Xi(\gamma), \quad \text{where}$$

$$\Xi(\gamma) = \Lambda_q^- \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_{h(\gamma_i, \gamma_{i+1})}(\gamma_{i+1}) - \overline{\psi}_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)}{\overline{\psi}_{h(\gamma_i, \gamma_{i+1})}(q) - \overline{\psi}_{h(\gamma_i, \gamma_{i+1})}(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} \quad \text{for all } \gamma \in C(p, q), \text{ and}$$

$$C(p, q) = \{\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1}) \in \Sigma(p, q) \mid h(\gamma_1, \gamma_2) \leq h(\gamma_2, \gamma_3) \leq \cdots \leq h(\gamma_{|\gamma|}, \gamma_{|\gamma|+1})\}.$$

2. Assume that the fiber F_j of p_j satisfies $H^*(F_j; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim(F_j)}; A)$ as rings for all j . Then for each path $\gamma \in C(p, q)$,

²In this paper, our convention is that an empty composition or product is the identity. Hence $\pi_k = \text{id}_{M_k}$.

- $\Xi(\gamma)$ can be written as the product of positive weights in ℓ^* and a constant C in A ; moreover, $C > 0$ if $\left\langle \frac{\alpha_r(r')}{\Lambda_{r'}}, \xi \right\rangle > 0$ for all $(r, r') \in E$.
- If (M, ω, ψ) is a GKM space, then $\Xi(\gamma)$ can be written as the product of distinct positive weights in $\Pi_q(M)$ and a constant C in A . Finally, if $\Theta(r, r') > 0$ for all $(r, r') \in E$, then $C > 0$; similarly, if $\Theta(r, r') \in A^\times$ for all $(r, r') \in E$, then $C \in A^\times$.

Remark 4.5. In fact, if M_k is a GKM space, then our proof demonstrates that the first claim holds whenever $p_j: M_{j+1} \rightarrow M_j$ is a weight preserving map for all j ; (see Definition 4.9).

Remark 4.6. If M_k has a discrete fixed set (or is a GKM space), then M_j has a discrete fixed set (or is a GKM space) for all j . To see this, consider any $q \in M_j^T$. Since the fiber $p_j^{-1}(q)$ is a Hamiltonian T -manifold, there exists $r \in M_{j+1}^T$ such that $p_j(r) = q$. Since the differential dp_j is surjective, the set of weights in the representation $T_q M_j$ is a subset of the set of weights in $T_r M_{j+1}$.

Theorem 4.4 has the following useful corollary.

Corollary 4.7. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds with discrete fixed sets, and let $\pi: M \rightarrow \widetilde{M}$ be a strong symplectic fibration. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. Fix p and q in M^T .

1. There exist canonical classes $\widehat{\alpha}_s \in H_T^{2\widehat{\lambda}(s)}(\widehat{M}_q; A)$ on the fiber $\widehat{M}_q = \pi^{-1}(\pi(q))$ for all $s \in \widehat{M}_q^T$.
2. Given $s \in \widehat{M}_q^T$, let $\overline{\Sigma}(p, s)$ denote the set of paths $\gamma = (\gamma_1, \dots, \gamma_{k+1})$ from p to s in the associated canonical graph (V, E) such that $\pi(\gamma_i) \neq \pi(\gamma_{i+1})$ for all i . Then

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p, s)} P(\gamma) \right) \widehat{\alpha}_s(q), \quad \text{where}$$

$$P(\gamma) = \widetilde{\Lambda}_{\pi(s)}^- \prod_{i=1}^{|\gamma|} \frac{\widetilde{\psi}(\pi(\gamma_{i+1})) - \widetilde{\psi}(\pi(\gamma_i))}{\widetilde{\psi}(\pi(s)) - \widetilde{\psi}(\pi(\gamma_i))} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} \quad \text{for all } s \in \widehat{M}_q^T \text{ and } \gamma \in \overline{\Sigma}(p, s).$$

3. Assume that $H^*(\widetilde{M}; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim(\widetilde{M})}; A)$ as rings. Then for all $s \in \widehat{M}_q^T$ and each path $\gamma \in \overline{\Sigma}(p, s)$
 - $P(\gamma)$ can be written as the product of positive weights in ℓ^* and a constant C in A ; moreover, $C > 0$ if $\left\langle \frac{\alpha_r(r')}{\Lambda_{r'}}, \xi \right\rangle > 0$ for all $(r, r') \in E$.
 - If (M, ω, ψ) is a GKM space, then $P(\gamma)$ can be written as the product of distinct positive weights in $\Pi_q(M)$ and a constant C in A . Finally, if $\Theta(r, r') > 0$ for all $(r, r') \in E$, then $C > 0$; similarly, if $\Theta(r, r') \in A^\times$ for all $(r, r') \in E$, then $C \in A^\times$.

If (M, ω, ψ) is a GKM space we will give an explicit description of $P(\gamma)$ in Lemma 4.23.

Proof of claim 1. of Theorem 4.4. We are now ready to begin the proof of the first part of our main theorem. We will begin with the special case of GKM spaces, where the

proof is easier and the main ideas are more transparent. However, the proof in the general case is self-contained; the reader may skip directly to that case.

The case of GKM spaces. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces, and let (V, E_{GKM}) and $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ be the associated GKM graphs. If $\pi: M \rightarrow \widetilde{M}$ is an equivariant map, the following statements hold:

- Given a vertex $p \in V$, $\pi(p) \in \widetilde{V}$.
- Given an edge $e = (p, q) \in E_{\text{GKM}}$, either $\pi(p) = \pi(q) \in \widetilde{V}$ or $\pi(e) = (\pi(p), \pi(q)) \in \widetilde{E}_{\text{GKM}}$ and $\eta(\pi(e))$ is a non-zero multiple of $\eta(e)$.

To see this, let $K \subset T$ be the maximal subgroup so that p and q are contained in the same connected component $N \subset M^K$. Since π is equivariant, either $\pi(N)$ is a fixed point in \widetilde{M} , or $\pi(N)$ is the connected component of $\widetilde{M}^{K'}$ for some subgroup $K' \subsetneq T$ which contains K .

Definition 4.8. We will say that an edge $(p, q) \in (V, E_{\text{GKM}})$ is **horizontal** (with respect to π) if $\pi(p) \neq \pi(q)$; moreover, we will say that a path γ in (V, E_{GKM}) is **horizontal** if all its edges are horizontal.

If $\pi: M \rightarrow \widetilde{M}$ is an equivariant fiber bundle and $e \in E_{\text{GKM}}$ is a horizontal edge, then $\eta(e) = \pm \eta(\pi(e))$. However, this need not hold for arbitrary equivariant maps.

Definition 4.9. We will say that a map $\pi: M \rightarrow \widetilde{M}$ is **weight preserving** if it is equivariant and $\eta(e) = \eta(\pi(e))$ for all horizontal edges $(p, q) \in E_{\text{GKM}}$.

Note that the composition of two weight preserving maps is itself weight preserving. In contrast, the composition of two strong symplectic fibrations may not be a strong symplectic fibration; indeed, it may not have symplectic fibers. However, the following assertion is clear; cf. Remark 4.3.

Lemma 4.10. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces. If $\pi: M \rightarrow \widetilde{M}$ is a strong symplectic fibration then π is weight preserving.

To prove claim 1., we need to check that the pull-back of a symplectic form and moment map by a weight preserving map satisfies criterion (1) of Corollary 2.4. We will do this in two steps.

Lemma 4.11. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces, and let $\pi: M \rightarrow \widetilde{M}$ be a weight preserving map. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Given a horizontal edge (p, q) in the GKM graph associated to M ,

$$\psi^\xi(q) - \psi^\xi(p) > 0 \quad \text{if and only if} \quad \widetilde{\psi}^\xi(\pi(q)) - \widetilde{\psi}^\xi(\pi(p)) > 0.$$

Proof. Since (p, q) is a horizontal edge and π is a weight preserving map, $\eta(\pi(p), \pi(q)) = \eta(p, q)$. Therefore, $\psi^\xi(q) - \psi^\xi(p)$ and $\widetilde{\psi}^\xi(\pi(q)) - \widetilde{\psi}^\xi(\pi(p))$ are both positive multiples of $\eta(p, q)$. \square

Lemma 4.12. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces, and let $\pi: M \rightarrow \widetilde{M}$ be a weight preserving map. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Given a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ at $p \in M^T$,

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \text{ such that } \pi(q) \neq \pi(p) \text{ and } \widetilde{\psi}^\xi(\pi(q)) \leq \widetilde{\psi}^\xi(\pi(p)).$$

Proof. Assume that $\alpha_p(q) \neq 0$ for some $q \in M^T$. By Lemma 1.9, there exists an ascending path γ from p to q in (V, E_{GKM}) . By Lemma 4.11 and the definition of ascending,

$$\tilde{\psi}^\xi(\pi(\gamma_i)) < \tilde{\psi}^\xi(\pi(\gamma_{i+1})) \quad \text{or} \quad \pi(\gamma_i) = \pi(\gamma_{i+1}) \quad \text{for each } i.$$

□

Proof of claim 1. of Theorem 4.4 in the case the M_j 's are GKM spaces. Let $w_j = \pi_j^*(\omega_j + \psi_j) \in H_T^2(M_k; \mathbb{R})$ for each $j \in \{1, \dots, k\}$. Since $\pi_k = \text{id}_{M_k}$, it is obvious that $w_k(r) \neq w_k(r')$ for all $(r, r') \in E_{\text{GKM}}$. By Lemma 4.10, each p_i is a weight preserving map, and so π_j is a weight preserving map for all j . Therefore, claim 1. of Theorem 4.4 is an immediate consequence of Corollary 2.4 and Lemma 4.12. □

The general case. The proof in the general case is nearly identical, except that it takes more work to prove Lemma 4.15, the analog of Lemma 4.12.

Lemma 4.13. *Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds, and let $\pi: M \rightarrow \widetilde{M}$ be a equivariant fiber bundle with symplectic fibers. For all sufficiently small $t > 0$,*

- (1) *the two-form $\omega_t = \pi^*(\widetilde{\omega}) + t\omega$ is symplectic; moreover,*
- (2) *as symplectic representations $(T_p M, \omega_t) \simeq (T_p \widehat{M}_p, \omega|_{\widehat{M}_p}) \oplus (T_{\pi(p)} \widetilde{M}, \widetilde{\omega})$ for all $p \in M^T$, where \widehat{M}_p denotes the fiber $\pi^{-1}(\pi(p))$.*

Proof. Let $V \subset TM$ be the kernel of the map $d\pi: TM \rightarrow T\widetilde{M}$. By assumption, π is a submersion; hence, $V \subset TM$ is a subbundle. Since we have assumed that π has symplectic fibers, the restriction $\omega|_V$ is symplectic. Since $\pi^*(\widetilde{\omega})|_V = 0$, this implies that the restriction $\omega_t|_V = t\omega|_V$ is symplectic and that $(V_p, \omega_t) \simeq (V_p, \omega)$ for all $p \in M^T$ and $t > 0$.

Let $H = V^\omega \subset TM$ be the symplectic perpendicular to V with respect to ω . Since $\pi^*(\widetilde{\omega})|_V = 0$, H is also symplectically perpendicular to V with respect to ω_t for all $t \geq 0$. Moreover, since $\omega|_V$ is symplectic, $H \subset TM$ is a subbundle and $TM = V \oplus H$. Thus, the map $d\pi: H \rightarrow T\widetilde{M}$ is an isomorphism. Since $\widetilde{\omega}$ is symplectic, this implies that the restriction $\pi^*(\widetilde{\omega})|_H$ is symplectic and that $(H_p, \pi^*(\widetilde{\omega})|_{H_p}) \simeq (T_{\pi(p)} \widetilde{M}, \widetilde{\omega})$ for all $p \in M^T$. Since being symplectic is an open condition and M is compact, analogous statements hold for ω_t for all sufficiently small $t > 0$. The claim follows immediately. □

Lemma 4.14. *Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set. Let $\varphi = \psi^\xi$ be a generic component of the moment map, and let $\overline{\varphi}: M \rightarrow \mathbb{R}$ be an invariant Morse-Bott function. Assume that for all $\epsilon > 0$ there exists a symplectic form $\omega' \in \Omega^2(M)$ with moment map ψ' such that:*

- (a) *$|(\psi')^\xi(x) - \overline{\varphi}(x)| < \epsilon$ for all $x \in M$; and*
- (b) *the product of the positive weights for the isotropy action of T on $(T_p M, \omega')$ is Λ_p^- for all $p \in M^T$.*

If $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ is the canonical class (with respect to φ) at $p \in M^T$, and \widehat{M}_p is the critical component of $\overline{\varphi}$ that contains p , then

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \text{ so that } q \notin \widehat{M}_p \text{ and } \overline{\varphi}(q) \leq \overline{\varphi}(p).$$

Moreover, the restriction of α_p to $\overline{\varphi}^{-1}(-\infty, \overline{\varphi}(p) - \delta)$ vanishes for all $\delta > 0$.

Proof. We may assume without loss of generality that $\overline{\varphi}(p) = 0$. By assumption, for any $\epsilon > 0$ there exists a symplectic form ω' with moment map ψ' such that (a) and (b) hold. Let φ' be $(\psi')^\xi$. Lemma 1.3 implies that

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \text{ such that } \varphi'(q) < \varphi'(p);$$

c.f. Lemma 3.5. By injectivity, this implies that the restriction of α_p to $(\varphi')^{-1}(-\infty, \varphi'(p))$ vanishes. Finally, (a) implies that

$$(\overline{\varphi})^{-1}(-\infty, -2\epsilon) \subset (\varphi')^{-1}(-\infty, \varphi'(p)),$$

and so the restriction of α_p to $\overline{\varphi}^{-1}(-\infty, -2\epsilon)$ vanishes.

Since $\overline{\varphi}$ is a Morse-Bott function there exists $\epsilon > 0$ so that 0 is the only critical value of $\overline{\varphi}$ in $[-2\epsilon, 2\epsilon]$. Since the restriction of α_p to $\overline{\varphi}^{-1}(-\infty, -2\epsilon)$ vanishes, the restriction of α_p to \widehat{M}_p is a multiple of the equivariant Euler class of the negative normal bundle of $\overline{\varphi}$ at \widehat{M}_p , and so there exists $\alpha'_p \in H_T^{2\lambda(p)}((\overline{\varphi})^{-1}(-\infty, 2\epsilon); A)$ so that $\alpha'_p|_{\widehat{M}_p} = \alpha_p|_{\widehat{M}_p}$, but $\alpha'_p|_C = 0$ for every other critical set C of $\overline{\varphi}$ so that $\overline{\varphi}(C) \leq 2\epsilon$. Moreover, since $\overline{\varphi}$ is invariant and the fixed set is discrete, every fixed point is critical. Hence,

$$(4) \quad \alpha'_p(q) = \alpha_p(q) \quad \text{for all } q \in \widehat{M}_p \cap M^T, \quad \text{and}$$

$$(5) \quad \alpha'_p(q) = 0 \quad \text{for all } q \in M^T \text{ such that } q \notin \widehat{M}_p \text{ and } \overline{\varphi}(q) \leq 2\epsilon.$$

By (a), $(\varphi')^{-1}(-\infty, \epsilon) \subset (\overline{\varphi})^{-1}(-\infty, 2\epsilon)$. Hence, we can restrict α'_p to $(\varphi')^{-1}(-\infty, \epsilon)$; moreover, this restriction satisfies (4) and

$$(6) \quad \alpha'_p(q) = 0 \quad \text{for all } q \in M^T \text{ such that } q \notin \widehat{M}_p \text{ and } \varphi'(q) < \epsilon.$$

By surjectivity, we can extend α'_p to a class (which we still call α'_p) on M with the same properties. Moreover, by the definition of canonical class,

$$\alpha_p(q) = 0 \quad \text{for all } q \in M^T \setminus \{p\} \text{ such that } \lambda(q) \leq \lambda(p).$$

Therefore, by (4) and (6),

$$\alpha_p(q) = \alpha'_p(q) \quad \text{for all } q \in M^T \text{ such that } \varphi'(q) < \epsilon \text{ and } \lambda(q) \leq \lambda(p).$$

Assume that there exists $r \in M^T$ such that $\alpha_p(r) \neq \alpha'_p(r)$ and $\varphi'(r) < \epsilon$ but $\alpha_p(s) = \alpha'_p(s)$ for all $s \in M^T$ such that $\varphi'(s) < \varphi'(r)$. By the equation above, this implies that $\lambda(r) > \lambda(p)$. Since $\beta = \alpha_p - \alpha'_p$ has degree $2\lambda(p)$, this contradicts Lemma 1.5. Therefore,

$$\alpha_p(q) = \alpha'_p(q) \quad \text{for all } q \in M^T \text{ such that } \varphi'(q) < \epsilon.$$

Finally, since $(\overline{\varphi})^{-1}((-\infty, 0]) \subset (\varphi')^{-1}(-\infty, \epsilon)$ by (a), this implies that

$$\alpha_p(q) = \alpha'_p(q) \quad \text{for all } q \in M^T \text{ such that } \overline{\varphi}(q) \leq 0.$$

Therefore, the claim follows immediately from (5). \square

Lemma 4.15. *Let $\{(M_j, \omega_j, \psi_j)\}_{j=1}^n$ be Hamiltonian T -manifolds with discrete fixed sets and let $\{p_j: M_{j+1} \rightarrow M_j\}_{j=1}^{n-1}$ be strong symplectic fibrations. Let $\varphi_n = \psi_n^\xi$ be a generic component of the moment map. Let $\pi = p_1 \circ p_2 \circ \cdots \circ p_{n-1}: M_n \rightarrow M_1$. Given a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M_n; A)$ at $p \in M_n^T$,*

$$\alpha_p(q) = 0 \quad \text{for all } q \in M_n^T \text{ so that } \pi(q) \neq \pi(p) \text{ and } \psi_1^\xi(\pi(q)) \leq \psi_1^\xi(\pi(p)).$$

Proof. Given $q \in M_n^T$, let $q_j = (p_j \circ p_{j+1} \circ \cdots \circ p_{n-1})(q) \in M_j$ for all j , let $X_j = p_{j-1}^{-1}(q_{j-1}) \subset M_j$ for $j > 1$, and let $X_1 = M_1$. By the definition of strong symplectic fibration and induction on n , as symplectic representations

$$(T_q M_n, \omega_n) \simeq (T_{q_1} X_1, \omega_1|_{X_1}) \oplus \cdots \oplus (T_{q_n} X_n, \omega_n|_{X_n}).$$

Therefore, by Lemma 4.13 and induction on n , for any $\epsilon > 0$ there exists a symplectic form $\omega'_n \in \Omega^2(M_n)$ with moment map ψ'_n such that:

- $|(\psi'_n)^\xi(x) - \pi^*(\psi_1)^\xi(x)| < \epsilon$ for all $x \in M_n$; and
- as symplectic representations $(T_q M_n, \omega_n) \simeq (T_q M_n, \omega'_n)$ for all $q \in M_n^T$.

Since M_1 has a discrete fixed set, $\psi_1^\xi: M_1 \rightarrow \mathbb{R}$ is a Morse function on M_1 with critical set M_1^T . Since π is an equivariant fiber bundle, this implies that $\pi^*(\psi_1)^\xi$ is an invariant Morse-Bott function on M , and that the critical component of $\pi^*(\psi_1)^\xi$ that contains $p \in M_n^T$ is the fiber $\pi^{-1}(\pi(p))$. Therefore, the claim follows from Lemma 4.14. \square

Proof of claim 1. of Theorem 4.4 in the general case. Let $w_j = \pi_j^*(\omega_j + \psi_j) \in H_T^2(M_k; \mathbb{R})$ for each $j \in \{1, \dots, k\}$. Since $\pi_k = \text{id}_{M_k}$, Lemma 1.3 implies that $w_k(r) \neq w_k(r')$ for all $(r, r') \in E$. Therefore, claim 1. of Theorem 4.4 is an immediate consequence of Corollary 2.4 and Lemma 4.15. \square

Proof of claim 2. of Theorem 4.4. Let a subtorus $T \subset SU(n+1)$ act on (\mathbb{CP}^n, ω) , and let φ be a generic component of the moment map $\psi: \mathbb{CP}^n \rightarrow \mathbb{R}$. If $[\omega]$ generates $H^2(\mathbb{CP}^n; \mathbb{Z})$ then

$$\Lambda_p^- = \prod_{\varphi(y) < \varphi(p)} \psi(p) - \psi(y),$$

where the sum is over all $y \in (\mathbb{CP}^n)^T$ such that $\varphi(y) < \varphi(p)$. The next lemma, which is the key ingredient in the proof of claim 2. of Theorem 4.4 and claim 3. of Corollary 4.7, generalizes this fact to other manifolds with isomorphic cohomology rings.

Definition 4.16. Let (M, ω, ψ) be a GKM space with GKM graph (V, E_{GKM}) . The **magnitude** of an edge $(r, s) \in E_{\text{GKM}}$ is

$$m(r, s) = \frac{\psi(s) - \psi(r)}{\eta(r, s)}.$$

Lemma 4.17. Let (M, ω, ψ) be a Hamiltonian T -manifold with discrete fixed set, and let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that $[\omega]$ generates $H^2(M; \mathbb{Z})$ and that $H^*(M; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim M}; A)$ as rings. Given $p \in M^T$, fix a subset $S \subset \{y \in M^T \mid \varphi(y) < \varphi(p)\}$. Then

- $\Lambda_p^- \prod_{y \in S} \frac{1}{\psi(p) - \psi(y)}$ can be written as the product of positive weights in ℓ^* and a constant in A_+ .
- If (M, ω, ψ) is a GKM space with GKM graph (V, E_{GKM}) , then $m(r, s)$ is a unit in A_+ for all $(r, s) \in E_{\text{GKM}}$ such that $\varphi(r) < \varphi(s)$. In particular $\Lambda_p^- \prod_{y \in S} \frac{1}{\psi(p) - \psi(y)}$ can be written as the product of distinct positive weights in $\Pi_p(M)$ and a unit in A_+ .

Proof. Since the fixed set is discrete and φ is a perfect Morse function, there is exactly one fixed point of index $2i$ for all $i \in \{0, \dots, \frac{1}{2} \dim(M)\}$. Therefore, there are exactly $\lambda(p)$ fixed points y with $\lambda(y) < \lambda(p)$. Moreover, by [LT, Lemma 2.7], the fact that $[\omega]$ is

integral implies that $[\omega + \psi - \psi(y)] \in H_T^2(M; \mathbb{Z})$ for all $y \in M^T$. Therefore, we may define a class

$$\beta = \prod_{\lambda(y) < \lambda(p)} [\omega + \psi - \psi(y)] \in H_T^{2\lambda(p)}(M; \mathbb{Z}),$$

where the product is over all $y \in M^T$ such that $\lambda(y) < \lambda(p)$.

Since $H^{2i}(M; \mathbb{R}) = H^{2i}(\mathbb{CP}^{\frac{1}{2}\dim(M)}; \mathbb{R})$ for all i , [T, Proposition 3.4] (and the fact that rational $\xi \in \mathfrak{t}$ are dense) implies that

$$(7) \quad \varphi(y) < \varphi(p) \quad \text{exactly if} \quad \lambda(y) < \lambda(p) \quad \text{for all } y \in M^T.$$

Since $\beta(y) = 0$ for all $y \in M^T$ such that $\lambda(y) < \lambda(p)$, Lemma 1.4, Corollary 1.5, and (7) together imply that we can write

$$\beta = \sum_{\lambda(y) \geq \lambda(p)} x_y \gamma_y,$$

where the sum is over $y \in M^T$ such that $\lambda(y) \geq \lambda(p)$, $\gamma_y \in H_T^{2\lambda(y)}(M; \mathbb{Z})$, $x_y \in H^{2\lambda(p)-2\lambda(y)}(BT; \mathbb{Z})$ for all $y \in M^T$, and $\{\gamma_y\}_{y \in M^T}$ is a basis for $H_T^*(M; \mathbb{Z})$ as a $H^*(BT; \mathbb{Z})$ module. Since p is the only fixed point with index $2\lambda(p)$, by degree considerations this implies that

$$(8) \quad \beta = x_p \gamma_p, \quad \text{where } x_p \in \mathbb{Z}.$$

Since $[\omega]^{\lambda(p)}$ is the image of β under the natural restriction map from $H_T^*(M; \mathbb{Z})$ to $H^*(M; \mathbb{Z})$, this implies that

$$[\omega]^{\lambda(p)} = x_p \tilde{\gamma}_p \quad \text{for all } p \in M^T,$$

where $\tilde{\gamma}_p$, the restriction of γ_p , generates $H^{2\lambda(p)}(M; \mathbb{Z})$. Moreover, since we have assumed that $[\omega]$ generates $H^2(M; \mathbb{Z})$ and that $H^*(M; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2}\dim M}; A)$ as rings, $[\omega]^{\lambda(p)}$ generates $H^{2\lambda(p)}(M; A)$. Hence the equation above implies that x_p must be invertible in A . Therefore, evaluating both sides of (8) at p ,

$$(9) \quad \Lambda_p^- \prod_{\varphi(y) < \varphi(p)} \frac{1}{\psi(p) - \psi(y)} = \frac{1}{x_p} \in A.$$

Now observe that $\psi(p) - \psi(y)$ is the product of a positive integer and a positive weight in ℓ^* for all $y \in M^T$ such that $\varphi(y) < \varphi(p)$. This proves the first claim.

If M is a GKM space, then since the GKM graph is $\frac{1}{2}\dim(M)$ -valent and has $\frac{1}{2}\dim(M) + 1$ vertices, it is a complete graph. Therefore $\psi(p) - \psi(y) = m(y, p)\eta(y, p)$ for all $y \in M^T$; moreover $m(y, p) \in \mathbb{Z}_+$ and $\eta(y, p)$ is a positive weight in $\Pi_p(M)$ for all $y \in M^T$ such that $\varphi(y) < \varphi(p)$. By (9) we have that

$$(10) \quad x_p = \prod_{\varphi(y) < \varphi(p)} m(y, p)$$

is a unit in A_+ , which implies that $m(y, p)$ is a unit in A_+ for all $y, p \in M^T$ such that $\varphi(y) < \varphi(p)$. Finally observe that the weights in $\Pi_p(M)$ are all distinct since M is a GKM space, and the second claim follows immediately. (Notice that in the GKM case (7) directly follows from the fact that the GKM graph is complete.) \square

Remark 4.18. If (M, ω, ψ) is a GKM space with GKM graph (V, E_{GKM}) , $H^*(M; \mathbb{R}) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim M}; \mathbb{R})$ and $[\omega]$ generates $H^2(M; \mathbb{Z})$, it follows from (10) that the magnitudes of the edges of (V, E_{GKM}) determine the ring structure of $H^*(M; \mathbb{Z})$.

We also need a technical lemma.

Lemma 4.19. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds, and let $\pi: M \rightarrow \widetilde{M}$ be a strong symplectic fibration. The natural restriction map $H_T^*(M; \mathbb{Z}) \rightarrow H^*(\widehat{M}_p; \mathbb{Z})$ is surjective for all $p \in M$, where \widehat{M}_p is the fiber $\pi^{-1}(\pi(p))$.

Proof. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Since $\widetilde{M} \times_T ET$ is connected and π induces a fiber bundle $M \times_T ET \rightarrow \widetilde{M} \times_T ET$ with fiber \widehat{M}_p , we may assume that $\pi(p) \in \widetilde{M}^T$ is the minimal fixed point. Hence, $\widetilde{\Lambda}_{\pi(p)}^- = 1$.

Consider any point $q \in \widehat{M}_p^T$. By Lemma 1.4, there exists a class $\gamma_q \in H^{2\lambda(q)}(M; \mathbb{Z})$ such that $\gamma_q(q) = \Lambda_q^-$ and $\gamma_q(r) = 0$ for all $r \in M^T \setminus \{q\}$ such that $\varphi(r) \leq \varphi(q)$. By definition of strong symplectic fibration and the paragraph above, $\Lambda_q^- = \widetilde{\Lambda}_{\pi(q)}^- \widehat{\Lambda}_q^- = \widetilde{\Lambda}_{\pi(p)}^- \widehat{\Lambda}_q^- = \widehat{\Lambda}_q^-$. Therefore, if $\widehat{\beta}_q$ denotes the restriction of γ_q to \widehat{M}_p , then $\widehat{\beta}_q(q) = \widehat{\Lambda}_q^-$, and $\widehat{\beta}_q(r) = 0$ for all $r \in \widehat{M}_p^T \setminus \{q\}$ such that $\varphi(r) \leq \varphi(q)$. By Lemma 1.4, this implies that $\{\widehat{\beta}_q\}_{q \in \widehat{M}_p^T}$ is a basis for $H_T^*(\widehat{M}_p; \mathbb{Z})$ as a $H^*(BT; \mathbb{Z})$ module. Hence, the map $H_T^*(M; \mathbb{Z}) \rightarrow H_T^*(\widehat{M}_p; \mathbb{Z})$ is surjective. Finally, since the fixed set is discrete, $H^*(M^T; \mathbb{Z})$ is torsion-free, and so the natural restriction map $H_T^*(\widehat{M}_p; \mathbb{Z}) \rightarrow H^*(\widehat{M}_p; \mathbb{Z})$ is surjective; see, for example, [T, §2]. \square

Proof of claim 2. of Theorem 4.4. Let $q_j = \pi_j(q)$ for all j , and let $X_j = p_{j-1}^{-1}(q_{j-1}) \subset M_j$ be the fiber over q_{j-1} . Note that the value of $\Xi(\gamma)$ doesn't change if we multiply $\omega_j + \psi_j$ by a non-zero constant or add any constant to it. Moreover, by Lemma 4.19 the restriction map from $H_T^2(M_j; \mathbb{Z})$ to $H^2(X_j; \mathbb{Z})$ is surjective. Therefore, since $H^2(X_j; \mathbb{R}) = \mathbb{R}$, we may assume that $[\omega_j + \psi_j]$ lies in³ $H_T^2(M_j; \mathbb{Z})$ and that $[\omega_j|_{X_j}]$ generates $H^2(X_j; \mathbb{Z})$.

Let $\widehat{\Lambda}_{q_j}^-$ denote the equivariant Euler class of the negative normal bundle of $\psi_j^\xi|_{X_j}$ at $q_j \in X_j$, and let $\Lambda_{q_j}^-$ denote the equivariant Euler class of the negative normal bundle of ψ_j^ξ at $q_j \in M_j$. By the definition of strong symplectic fibration, $\Lambda_{q_j}^- = \widehat{\Lambda}_{q_j}^- \Lambda_{q_{j-1}}^-$ for all j . Since M_0 is a point, this implies by induction that

$$\Lambda_q^- = \prod_{j=1}^k \widehat{\Lambda}_{q_j}^-.$$

Therefore, to prove the claim it is enough to prove that given $h \in \{1, \dots, k\}$ such that the fiber X_h satisfies $H^*(X_h; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim X_h}; A)$ as rings, r and s in M_k^T such that $\pi_h(s) = \pi_h(q) = q_h$, and a path γ from r to s such that $h(\gamma_i, \gamma_{i+1}) = h$ for all $i \in \{1, \dots, |\gamma|\}$, if we define

$$\Xi_h(\gamma) = \widehat{\Lambda}_{q_h}^- \prod_{i=1}^{|\gamma|} \frac{\overline{\psi}_h(\gamma_{i+1}) - \overline{\psi}_h(\gamma_i)}{\overline{\psi}_h(q) - \overline{\psi}_h(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}$$

³Since the fixed set is discrete $H_T^2(M_j; \mathbb{Z})$ and $H^2(X_j; \mathbb{Z})$ are torsion-free. Therefore, we can identify these groups with their images in $H_T^2(M_j; \mathbb{R})$ and $H^2(X_j; \mathbb{R})$, respectively.

then

- (a1) $\Xi_h(\gamma)$ can be written as the product of positive weights in ℓ^* and a constant C in A ; moreover, $C > 0$ if $\left\langle \frac{\alpha_r(r')}{\Lambda_{r'}^-}, \xi \right\rangle > 0$ for all $(r, r') \in E$.
- (b1) If (M_k, ω_k, ψ_k) is a GKM space, then $\Xi_h(\gamma)$ can be written as the product of distinct positive weights in $\Pi_q(M)$ and a constant C in A . Finally, if $\Theta(r, r') > 0$ for all $(r, r') \in E$, then $C > 0$; similarly, if $\Theta(r, r') \in A^\times$ for all $(r, r') \in E$, then $C \in A^\times$.

To prove this, first note that since $h(\gamma_i, \gamma_{i+1}) = h$ for all i and $\pi_h(s) = \pi_h(q)$, $\pi_h(\gamma_i) \in X_h$ and $\pi_h(\gamma_i) \neq \pi_h(\gamma_{i+1})$ for all i . So by Lemma 4.15 (or Lemmas 4.10 and 4.12 if M_k is GKM)

$$(11) \quad \overline{\psi}_h^\xi(\gamma_i) < \overline{\psi}_h^\xi(\gamma_{i+1}) \quad \text{for all } 1 \leq i \leq |\gamma|.$$

Hence, $\pi_h(\gamma_i) \neq \pi_h(\gamma_j)$ for all $i \neq j$ and $\overline{\psi}_h^\xi(\gamma_i) < \overline{\psi}_h^\xi(s)$ for all $1 \leq i \leq |\gamma|$. Therefore, since $[\omega_j|_{X_j}]$ generates $H^2(X_j; \mathbb{Z})$ and $H^*(X_h; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim X_h}; A)$, Lemma 4.17 implies that

- (a2) $\widehat{\Lambda}_{q_h}^- \prod_{i=1}^{|\gamma|} \frac{1}{\overline{\psi}_h(q) - \overline{\psi}_h(\gamma_i)}$ can be written as the product of positive weights in ℓ^* and a constant in A_+ .
- (b2) If (M_k, ω_k, ψ_k) is a GKM space then $\widehat{\Lambda}_{q_h}^- \prod_{i=1}^{|\gamma|} \frac{1}{\overline{\psi}_h(q) - \overline{\psi}_h(\gamma_i)}$ can be written as the product of distinct positive weights in $\Pi_q(M_k)$ and a unit in A_+ .

Here, in the case that M_k is a GKM space, we use the fact that by Remark 4.6 M_j is also a GKM space for all j ; moreover by Lemma 4.10 p_j is a weight preserving map for all j , hence π_h is weight preserving as well and $\Pi_{q_h}(X_h) \subset \Pi_{q_h}(M_h)$ is a subset of $\Pi_q(M_k)$.

Since $[\omega_h + \psi_h]$ is an integral class, Lemma 2.3 and (11) together imply that for all $1 \leq i \leq |\gamma|$,

$$(12) \quad (\overline{\psi}_h(\gamma_{i+1}) - \overline{\psi}_h(\gamma_i)) \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} \in \begin{cases} A, & \text{and} \\ A_+ & \text{if } \left\langle \frac{\alpha_r(r')}{\Lambda_{r'}^-}, \xi \right\rangle > 0 \ \forall (r, r') \in E. \end{cases}$$

If M_k is a GKM space then by Theorem 1.10, $\frac{\alpha_r(r')}{\Lambda_{r'}^-} = \frac{\Theta(r, r')}{\eta(r, r')}$ for all $(r, r') \in E$. Moreover Lemma 1.3 implies that $\psi_k^\xi(r) < \psi_k^\xi(r')$ and so, since $\psi_k(r') - \psi_k(r)$ is a positive multiple of $\eta(r, r')$, $\langle \eta(r, r'), \xi \rangle > 0$. Therefore,

$$(13) \quad \text{if } M_k \text{ is GKM, then } \left\langle \frac{\alpha_r(r')}{\Lambda_{r'}^-}, \xi \right\rangle > 0 \text{ exactly if } \Theta(r, r') > 0 \text{ for all } (r, r') \in E.$$

Moreover, since π_h is a weight preserving map, $\frac{\overline{\psi}_h(\gamma_{i+1}) - \overline{\psi}_h(\gamma_i)}{\eta(\gamma_i, \gamma_{i+1})} = m(\pi_h(\gamma_i), \pi_h(\gamma_{i+1}))$. Hence by Lemma 4.17 we have that

$$(14) \quad (\overline{\psi}_h(\gamma_{i+1}) - \overline{\psi}_h(\gamma_i)) \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} \text{ is a unit in } A_+ \text{ exactly if } \Theta(\gamma_i, \gamma_{i+1}) \text{ is a unit in } A_+.$$

The claim now follows from (a2), (b2), (12), (13) and (14). \square

Proof of Corollary 4.7. The proof uses the following lemma.

Lemma 4.20. *Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds with discrete fixed sets, and let $\pi: M \rightarrow \widetilde{M}$ be a strong symplectic fibration. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Given $q \in M^T$, consider the fiber $\widehat{M}_q = \pi^{-1}(\pi(q))$. If $\alpha_s \in H^{2\lambda(s)}(M; A)$ is a canonical class at $s \in \widehat{M}_q^T$, then there exists $\widehat{\alpha}_s \in H_T^*(\widehat{M}_q; A)$ such that*

$$\widetilde{\Lambda}_{\pi(q)}^- \widehat{\alpha}_s = \alpha_s|_{\widehat{M}_q}.$$

Proof. Define $\overline{\varphi} = \pi^*(\widetilde{\psi})^\xi: M \rightarrow \mathbb{R}$. Since \widetilde{M} has a discrete fixed set, $\widetilde{\psi}^\xi: \widetilde{M} \rightarrow \mathbb{R}$ is a Morse function with critical set \widetilde{M}^T . Since π is a fiber bundle, this implies that $\overline{\varphi}$ is an invariant Morse-Bott function on M and that the critical component of $\overline{\varphi}$ that contains q is the fiber \widehat{M}_q . Moreover, the index of $\overline{\varphi}$ at \widehat{M}_q is $\widetilde{\lambda}(\pi(q))$, and the equivariant Euler class of the negative normal bundle of $\overline{\varphi}$ at \widehat{M}_q is $\widetilde{\Lambda}_{\pi(q)}^-$. By the definition of strong symplectic fibration, Lemma 4.13 and Lemma 4.14 imply that for any $s \in \widehat{M}_q^T$ the restriction of α_s to $\overline{\varphi}^{-1}(-\infty, \overline{\varphi}(q) - \delta)$ vanishes for all $\delta > 0$. Thus, by a standard Morse theory argument, there exists $\widehat{\alpha}_s \in H_T^{2\lambda(s) - 2\widetilde{\lambda}(\pi(q))}(\widehat{M}_q; A)$ such that $\widetilde{\Lambda}_{\pi(q)}^- \widehat{\alpha}_s = \alpha_s|_{\widehat{M}_q}$. \square

Proof of Corollary 4.7. Since π is a strong symplectic fibration, $\Lambda_s^- = \widetilde{\Lambda}_{\pi(s)}^- \widehat{\Lambda}_s^- = \widetilde{\Lambda}_{\pi(q)}^- \widehat{\Lambda}_s^-$ and $\lambda(s) = \widetilde{\lambda}(\pi(q)) + \widehat{\lambda}(s)$ for all $s \in \widehat{M}_q^T$. Hence, $\widehat{\lambda}(r) \leq \widehat{\lambda}(s)$ exactly if $\lambda(r) \leq \lambda(s)$ for all r and s in \widehat{M}_q^T .

By Lemma 4.20, for all $s \in \widehat{M}_q^T$ there exists a class $\widehat{\alpha}_s \in H_T^{2\widehat{\lambda}(s)}(\widehat{M}_q; A)$ such that $\widetilde{\Lambda}_{\pi(q)}^- \widehat{\alpha}_s = \alpha_s|_{\widehat{M}_q}$. Since $\alpha_s \in H_T^{2\lambda(s)}(M; A)$ is a canonical class, the paragraph above implies that $\widehat{\alpha}_s$ is a canonical class at s on \widehat{M}_q with respect to the restriction $\varphi|_{\widehat{M}_q}$. This proves the first claim. Moreover, applying Theorem 2.1 (and Remark 2.2) to \widehat{M}_q , we have

$$(15) \quad \widehat{\alpha}_s(q) = \widehat{\Lambda}_q^- \sum_{\gamma \in \widehat{\Sigma}(s, q)} \prod_{i=1}^{|\gamma|} \frac{\psi(\gamma_{i+1}) - \psi(\gamma_i)}{\psi(q) - \psi(\gamma_i)} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} \quad \text{for all } s \in \widehat{M}_q^T,$$

where $\widehat{\Sigma}(s, q)$ is the set of paths from s to q in the canonical graph associated to \widehat{M}_q .

Now we can apply Theorem 4.4 to $\pi: M \rightarrow \widetilde{M}$. Observe that a path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ from p to q lies in $C(p, q)$ exactly if there exists $j \in \{1, \dots, |\gamma|+1\}$ such that $\pi(\gamma_i) \neq \pi(\gamma_{i+1})$ for all $i < j$ and $\pi(\gamma_i) = \pi(\gamma_{i+1})$ for all $i \geq j$, that is, so that $(\gamma_1, \dots, \gamma_j)$ belongs to $\overline{\Sigma}(p, \gamma_j)$, and $(\gamma_j, \dots, \gamma_{|\gamma|+1})$ belongs to $\widehat{\Sigma}(\gamma_j, q)$. Hence, since $\Lambda_q^- = \widehat{\Lambda}_q^- \widetilde{\Lambda}_{\pi(q)}^-$, the second claim follows immediately from (15) and Theorem 4.4.

Finally, the third claim follows from (a1) and (b1). \square

Remark 4.21. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be Hamiltonian T -manifolds with discrete fixed sets, and let $\pi: M \rightarrow \widetilde{M}$ be a strong symplectic fibration. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that canonical classes $\alpha_p \in H_T^{2\lambda(p)}(M; A)$ exist for all $p \in M^T$. If (M, ω, ψ) is a GKM space, it is easy to see that there exist canonical

classes $\widehat{\alpha}_s \in H_T^{2\widehat{\lambda}(s)}(\widehat{M}_q; A)$ on the fiber $\widehat{M}_q = \pi^{-1}(\pi(q))$ with respect to the restriction $\varphi|_{\widehat{M}_q}$, for all $s, q \in \widehat{M}_q^T$.

In fact, since π is a strong symplectic fibration, $(\widehat{M}_q, \omega|_{\widehat{M}_q}, \psi|_{\widehat{M}_q})$ is a GKM space for all $q \in M^T$, and its GKM graph is just the restriction of the GKM graph of M to \widehat{M}_q . By Remark 4.3 in [GT], φ is index increasing on M . Since π is a strong symplectic fibration, $\lambda(s) - \lambda(r) = \widehat{\lambda}(s) - \widehat{\lambda}(r)$ for all $r, s \in \widehat{M}_q^T$; so $\varphi|_{\widehat{M}_q}$ is also index increasing. Therefore the claim follows from Theorem 1.10.

In our final lemma, we show how to express the polynomials $P(\gamma)$ appearing in Corollary 4.7 in terms of the magnitudes of the edges of the GKM graph associated to the base; see Definition 4.16.

Definition 4.22. Let $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be a GKM space with GKM graph $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$, and let $\widetilde{\psi}^\xi$ be a generic component of the moment map. Given an ascending path $\widetilde{\gamma} = (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_{|\widetilde{\gamma}|+1})$, the set of **skipped vertices** of $\widetilde{\gamma}$ is defined to be

$$SV(\widetilde{\gamma}) = \left\{ r \in \widetilde{V} \mid \widetilde{\psi}^\xi(r) < \widetilde{\psi}^\xi(\widetilde{\gamma}_{|\widetilde{\gamma}|+1}) \right\} \setminus \{\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_{|\widetilde{\gamma}|+1}\}.$$

Lemma 4.23. Let (M, ω, ψ) and $(\widetilde{M}, \widetilde{\omega}, \widetilde{\psi})$ be GKM spaces with GKM graphs (V, E_{GKM}) and $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ and let $\pi: M \rightarrow \widetilde{M}$ be a strong symplectic fibration. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Assume that φ is index increasing. Also assume that $H^*(\widetilde{M}; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim(\widetilde{M})}; A)$.

Given p and $s \in M^T$ and a horizontal path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ from p to s in the canonical graph (V, E) , define

$$P(\gamma) = \widetilde{\Lambda}_{\pi(s)}^- \prod_{i=1}^{|\gamma|} \frac{\widetilde{\psi}(\pi(\gamma_{i+1})) - \widetilde{\psi}(\pi(\gamma_i))}{\widetilde{\psi}(\pi(s)) - \widetilde{\psi}(\pi(\gamma_i))} \frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-}.$$

Then

$$P(\gamma) = \prod_{i=1}^{|\gamma|} \frac{m(\pi(\gamma_i), \pi(\gamma_{i+1})) \Theta(\gamma_i, \gamma_{i+1})}{m(\pi(\gamma_i), \pi(s))} \prod_{r \in SV(\pi(\gamma))} \eta(r, \pi(s)).$$

Proof. Observe that by Theorem 1.10, canonical classes α_p exist for all $p \in M^T$ and $(V, E) \subset (V, E_{\text{GKM}})$. By Lemma 4.10, π is weight preserving; hence $\eta(\gamma_i, \gamma_{i+1}) = \eta(\pi(\gamma_i), \pi(\gamma_{i+1}))$ for all i , and by the definition of magnitude $\widetilde{\psi}(\pi(\gamma_{i+1})) - \widetilde{\psi}(\pi(\gamma_i)) = \eta(\gamma_i, \gamma_{i+1}) m(\pi(\gamma_i), \pi(\gamma_{i+1}))$.

Since $H^*(\widetilde{M}; A) \simeq H^*(\mathbb{CP}^{\frac{1}{2} \dim(\widetilde{M})}; A)$, $(\widetilde{V}, \widetilde{E}_{\text{GKM}})$ is a complete graph (see the proof of Lemma 4.17), and so $\widetilde{\psi}(\pi(s)) - \widetilde{\psi}(\pi(\gamma_i)) = \eta(\pi(\gamma_i), \pi(s)) m(\pi(\gamma_i), \pi(s))$ for all $i \leq |\gamma|$. Moreover, by Lemma 1.3, γ is an ascending path; hence Lemma 4.11 implies that $\pi(\gamma)$

is ascending as well (with respect to $\widetilde{\psi}^\xi$), and so $\prod_{i=1}^{|\gamma|} \frac{\widetilde{\Lambda}_{\pi(s)}^-}{\eta(\pi(\gamma_i), \pi(s))} = \prod_{r \in SV(\pi(\gamma))} \eta(r, \pi(s))$.

Finally observe that by Theorem 1.10 $\frac{\alpha_{\gamma_i}(\gamma_{i+1})}{\Lambda_{\gamma_{i+1}}^-} = \frac{\Theta(\gamma_i, \gamma_{i+1})}{\eta(\gamma_i, \gamma_{i+1})}$ for all i . \square

5. COADJOINT ORBITS

We are now ready to apply our results to the important special case of coadjoint orbits.

Let G be a compact simple Lie group with Lie algebra \mathfrak{g} , and let $T \subset G$ be a maximal torus with Lie algebra \mathfrak{t} . Let $R \subset \mathfrak{t}^*$ denote the set of roots and W the Weyl group of G . Let $\langle \cdot, \cdot \rangle$ be a positive definite symmetric bilinear form on \mathfrak{g} which is G -invariant; we use it to embed \mathfrak{t}^* in \mathfrak{g}^* .

Given a point $p_0 \in \mathfrak{t}^*$, consider the coadjoint orbit $\mathcal{O}_{p_0} = G \cdot p_0$. Let $P_{p_0} \subset G$ be the stabilizer of p_0 ; the map which takes $g \in G$ to $g \cdot p_0 \in \mathcal{O}_{p_0}$ induces an identification $\mathcal{O}_{p_0} = G/P_{p_0}$. Finally, there is natural G -invariant complex structure J and a compatible symplectic form ω (the Kostant-Kirillov form) on \mathcal{O}_{p_0} ; the moment map is the inclusion map $\mathcal{O}_{p_0} \hookrightarrow \mathfrak{g}^*$. Hence, the moment map $\psi: \mathcal{O}_{p_0} \rightarrow \mathfrak{t}^*$ for the T action is the composition of this inclusion with the natural projection from \mathfrak{g}^* to \mathfrak{t}^* . Finally, $(\mathcal{O}_{p_0}, \omega, \psi)$ is a GKM space. (See [GHZ].)

Generic coadjoint orbits. Now fix a **generic** coadjoint orbit \mathcal{O}_{p_0} , that is, assume that $p_0 \in \mathfrak{t}^*$ lies in the interior of a Weyl chamber. The main goal of this subsection is to give an explicit description of the associated canonical graph.

Proposition 5.1. *Let the maximal torus T of a compact simple Lie group G act on a generic coadjoint orbit $\mathcal{O}_{p_0} \subset \mathfrak{g}^*$ with moment map $\psi: \mathcal{O}_{p_0} \rightarrow \mathfrak{t}^*$. Let $\varphi = \psi^\xi$ be a generic component of the moment map that achieves its minimal value at $p_0 \in \mathfrak{t}^*$, and let $R^+ = \{\alpha \in R \mid \langle \alpha, \xi \rangle > 0\}$.*

There exist canonical classes $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{p_0}; \mathbb{Z})$ for all $p \in \mathcal{O}_{p_0}^T$. Under the identification of the Weyl group W with $\mathcal{O}_{p_0}^T$ given by $w \mapsto w(p_0)$, the canonical graph is (W, E) , where

$$E = \{(\sigma, \sigma s_\beta) \in W \times W \mid l(\sigma s_\beta) = l(\sigma) + 1 \text{ and } \beta \in R^+\}; \quad \text{moreover,}$$

$$\alpha_{\sigma(p_0)}(\sigma'(p_0)) = \frac{\Lambda_{\sigma'(p_0)}^-}{\eta(\sigma, \sigma')} = \frac{\Lambda_{\sigma'(p_0)}^-}{\sigma(\beta)} \quad \text{for all } (\sigma, \sigma') \in E, \text{ where } \sigma' = \sigma s_\beta \text{ and } \beta \in R^+.$$

To prove this, first note that the GKM graph (V, E_{GKM}) of the coadjoint orbit \mathcal{O}_{p_0} can be described as follows:

- The map from the Weyl group W to \mathfrak{t}^* which takes w to $w(p_0)$ induces a bijection between the elements of the Weyl group and the vertices $V = \mathcal{O}_{p_0}^T$. The moment map ψ is the inclusion map, that is, $\psi(p) = p$ for all $p \in V$.
- There exist an edge $e \in E_{\text{GKM}}$ between two vertices $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$ if and only if $w_2 = s_\alpha w_1$, where s_α is the reflection associated to some $\alpha \in R$. In this case, the weight $\eta(p_1, p_2)$ is the unique $\alpha \in R$ such that $w_2 = s_\alpha w_1$ and $\langle p_2, \alpha \rangle > 0$.

In particular, the set of weights of the isotropy representation on $(T_p \mathcal{O}_{p_0}, \omega)$ is

$$(16) \quad \Pi_p(\mathcal{O}_{p_0}) = \{\alpha \in R \mid \langle p, \alpha \rangle > 0\} \quad \text{for all } p \in V.$$

Moreover, given $\alpha \in R$, $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$ in V such that $w_2 = s_\alpha w_1$

$$\psi(p_2) - \psi(p_1) = p_2 - s_\alpha(p_2) = 2 \frac{\langle p_2, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

Hence, as required, $\psi(p_2) - \psi(p_1)$ is a positive multiple of $\eta(p_1, p_2)$.

Now let $\varphi = \psi^\xi$ be a generic component of the moment map; assume that φ achieves its minimum value at p_0 . Let $R^+ = \{\alpha \in R \mid \langle \alpha, \xi \rangle > 0\}$ be the associated set of positive roots and $R_0 \subset R^+$ be the associated simple roots. Since p_0 is the minimum, $\langle \alpha, \xi \rangle < 0$ for every weight $\alpha \in \Pi_{p_0}(\mathcal{O}_{p_0})$. By (16), this implies that

$$(17) \quad \langle p_0, \alpha \rangle < 0 \quad \text{for all } \alpha \in R^+.$$

Remark 5.2. It is easy to check that

$$(18) \quad s_{w(\beta)}w = ws_\beta \quad \text{for all } w \in W \text{ and } \beta \in R.$$

Since the Weyl group takes R to itself, this implies that there exists an edge $e \in E_{\text{GKM}}$ between two vertices $p_1 = w_1(p_0)$ and $p_2 = w_2(p_0)$ if and only if $w_2 = w_1 s_\beta$ for some $\beta \in R^+$. In this case, since $\langle \cdot, \cdot \rangle$ is G -invariant, (17) implies that $\langle p_2, w_2(\beta) \rangle = \langle w_2(p_0), w_2(\beta) \rangle = \langle p_0, \beta \rangle < 0$. Therefore, $\eta(p_1, p_2) = -w_2(\beta) = -w_1 s_\beta(\beta) = w_1(\beta)$.

Moreover, $\langle w(p_0), \alpha \rangle = \langle p_0, w^{-1}(\alpha) \rangle$ for all $w \in W$, and so (16) and (17) together imply that $\Pi_{w(p_0)}(\mathcal{O}_{p_0}) = w(-R^+)$. Thus, since the set of weights $\Pi_p^-(\mathcal{O}_{p_0})$ in the negative normal bundle at p is the set of positive weights in the representation $(T_p \mathcal{O}_{p_0}, \omega)$,

$$(19) \quad \Pi_p^-(\mathcal{O}_{p_0}) = R^+ \cap w(-R^+) = -w(R^+ \cap w^{-1}(-R^+)) \quad \text{for all } p = w(p_0) \in V.$$

We will need the following standard facts about root systems [Hum]. First, every element w of the Weyl group W can be written as a product of simple reflections, i.e. $w = s_1 \cdots s_r$, where $s_i = s_{\alpha_i}$ and $\alpha_i \in R_0$ for all $i = 1, \dots, r$ [Hum, §1.5]. The **length** of w , denoted $l(w)$, is the smallest r for which such an expression exists. We refer to any such expression with $r = l(w)$ as a **reduced expression** for w .

1. Given $w \in W$ and $\beta \in R^+$, $l(ws_\beta) > l(w)$ exactly if $w(\beta) \in R^+$ [Hum, §5.7].
2. If $w = s_1 \cdots s_l$ is a reduced expression for $w \in W$, where $s_i = s_{\alpha_i}$ for some $\alpha_i \in R_0$ for all i , then (see [Hum, page 14])

$$R^+ \cap w^{-1}(-R^+) = \{\beta_1, \dots, \beta_l\} \quad \text{where } \beta_i = s_l \cdots s_{i+1}(\alpha_i).$$

Moreover, the β_i are distinct.

Combining (19) with fact 2. above, we see that for any $w \in W$ with reduced expression $w = s_1 \cdots s_l$,

$$(20) \quad \Pi_{w(p_0)}^-(\mathcal{O}_{p_0}) = \{\eta_1, \dots, \eta_l\} \quad \text{where } \eta_i = s_1 \cdots s_{i-1}(\alpha_i).$$

Therefore, since $\lambda(p) = |\Pi_p^-(\mathcal{O}_{p_0})|$,

$$(21) \quad \lambda(p) = l(w), \quad \text{for all } p = w(p_0) \in V.$$

By Theorem 1.10, the next lemma demonstrates that canonical classes exist on \mathcal{O}_{p_0} , thus proving the first claim of Proposition 5.1.

Lemma 5.3. *Let the maximal torus T of a compact simple Lie group G act on a generic coadjoint orbit $\mathcal{O}_{p_0} \subset \mathfrak{g}^*$ with moment map $\psi: \mathcal{O}_{p_0} \rightarrow \mathfrak{t}^*$. Then each generic component of the moment map, $\varphi = \psi^\xi$, is index increasing.*

Proof. Let (V, E_{GKM}) be the associated GKM graph. Assume that φ achieves its minimum value at $p_0 \in \mathfrak{t}^*$. Consider an edge $(p_1, p_2) = (w_1(p_0), w_2(p_0)) \in E_{\text{GKM}}$ so that $\varphi(p_2) > \varphi(p_1)$. By Remark 5.2, there exists $\beta \in R^+$ so that $w_2 = w_1 s_\beta$ and $\eta(p_1, p_2) = w_1(\beta)$. Since $\psi(p_2) - \psi(p_1)$ is a positive multiple of $\eta(p_1, p_2)$, the fact that $\varphi(p_2) > \varphi(p_1)$ implies that $w_1(\beta) \in R^+$. By fact 1. above, this implies that $l(w_2) = l(w_1 s_\beta) > l(w_1)$. Therefore, (21) implies that $\lambda(p_2) > \lambda(p_1)$, as required. \square

The explicit description of the canonical graph given in Proposition 5.1 follows immediately from Theorem 1.10, Remark 5.2, (21), and the following proposition, which describes the integers $\Theta(p, q)$ which appear in Theorem 1.10.

Proposition 5.4. *Let the maximal torus T of a compact simple Lie group G act on a generic coadjoint orbit $\mathcal{O}_{p_0} \subset \mathfrak{g}^*$ with moment map $\psi: \mathcal{O}_{p_0} \rightarrow \mathfrak{t}^*$. Let (V, E_{GKM}) be the associated GKM graph. Let $\varphi = \psi^\xi$ be a generic component of the moment map. Then*

$$\Theta(p, q) = 1 \quad \text{for all } (p, q) \in E_{\text{GKM}} \text{ with } \lambda(q) - \lambda(p) = 1.$$

Proof. Assume that φ achieves its minimum value at $p_0 \in \mathfrak{t}^*$. Let $p = w'(p_0)$ and $q = w(p_0)$ be fixed points such that $(p, q) \in E_{\text{GKM}}$ and $\lambda(q) - \lambda(p) = 1$. Let $\Pi_p^-(\mathcal{O}_{p_0})$ and $\Pi_q^-(\mathcal{O}_{p_0})$ denote the set of weights in the negative normal bundle of φ at p and q , respectively, and let $\alpha = \eta(p, q)$. In order to prove that $\Theta(p, q) = 1$, it is sufficient to find a bijection $f: \Pi_p^-(\mathcal{O}_{p_0}) \rightarrow \Pi_q^-(\mathcal{O}_{p_0}) \setminus \{\alpha\}$ such that for each $\eta \in \Pi_p^-(\mathcal{O}_{p_0})$, there exists a constant c such that $f(\eta) - \eta = c\alpha$.

Let $w = s_1 s_2 \cdots s_l$ be a reduced expression for w , where $s_i = s_{\alpha_i}$ for some $\alpha_i \in R_0$ for all i . Since $(p, q) \in E_{\text{GKM}}$, $w = s_\alpha w'$. Moreover, (21) implies that $l(w) = l(w') + 1 > l(w')$. Therefore, by the Strong Exchange Condition (cf. [Hum, Section 5.8]) $w' = s_1 \cdots \widehat{s_j} \cdots s_l$ for some (unique) j , where $\widehat{s_j}$ indicates that we are omitting the j 'th term. Let $\tilde{w} = s_1 s_2 \cdots s_{j-1}$. Then by (18) we have

$$s_1 s_2 \cdots s_k = \tilde{w} s_j s_{j+1} \cdots s_k = s_{\tilde{w}(\alpha_j)} \tilde{w} s_{j+1} \cdots s_k = s_{\tilde{w}(\alpha_j)} s_1 s_2 \cdots \widehat{s_j} \cdots s_k \quad \text{for all } j \leq k \leq l.$$

In particular, $w = s_{\tilde{w}(\alpha_j)} w'$, and so $s_{\tilde{w}(\alpha_j)} = s_\alpha$. Hence,

$$s_1 s_2 \cdots s_k (\alpha_{k+1}) \equiv s_1 s_2 \cdots \widehat{s_j} \cdots s_k (\alpha_{k+1}) \pmod{\alpha} \quad \text{for all } j \leq k < l.$$

Moreover, by (20), $\tilde{w}(\alpha_j) \in \Pi_q^-(M)$, and so

$$\begin{aligned} \Pi_q^- \setminus \{\alpha\} &= \{\alpha_1, s_1(\alpha_2), \dots, s_1 s_2 \cdots s_{j-2}(\alpha_{j-1}), s_1 \cdots s_j(\alpha_{j+1}), \dots, s_1 \cdots s_{l-1}(\alpha_l)\} \quad \text{and} \\ \Pi_p^- &= \{\alpha_1, s_1(\alpha_2), \dots, s_1 s_2 \cdots s_{j-2}(\alpha_{j-1}), s_1 \cdots s_{j-1}(\alpha_{j+1}), \dots, s_1 \cdots \widehat{s_j} \cdots s_{l-1}(\alpha_l)\}. \end{aligned}$$

The claim follows immediately. \square

Remark 5.5. Let w_1 and w_2 be two elements of the Weyl group W such that $l(w_1) < l(w_2)$ and $w_2 = w_1 s_\beta$, for some $\beta \in R^+$; in this case, we write $w_1 \rightarrow w_2$. The *Bruhat order* is the transitive closure of this order, i.e., $w < w'$ in the Bruhat order if there exists a sequence of elements of the Weyl group w_0, w_1, \dots, w_m such that $w_0 = w$, $w_m = w'$ and $w_i \rightarrow w_{i+1}$ for all $i = 0, \dots, m-1$. By Remark 5.2, (21), and Lemma 5.3, $w < w'$ exactly if there exists an ascending path from w to w' in (V, E_{GKM}) .

Maps between coadjoint orbits. Consider now two points p_0 and $\tilde{p}_0 \in \mathfrak{t}^*$ such that $P_{\tilde{p}_0} \supset P_{p_0}$, where P_{p_0} and $P_{\tilde{p}_0}$ are the stabilizers of p_0 and \tilde{p}_0 , respectively. Let \mathcal{O}_{p_0} and $\mathcal{O}_{\tilde{p}_0}$ be the coadjoint orbits through p_0 and \tilde{p}_0 , respectively, and let (V, E_{GKM}) and $(\tilde{V}, \tilde{E}_{\text{GKM}})$ be the GKM graphs associated to \mathcal{O}_{p_0} and $\mathcal{O}_{\tilde{p}_0}$, respectively. Since $\mathcal{O}_{p_0} = G/P_{p_0}$ and $\mathcal{O}_{\tilde{p}_0} \simeq G/P_{\tilde{p}_0}$, there is a natural projection map

$$\begin{aligned} \pi: \quad \mathcal{O}_{p_0} &\rightarrow \mathcal{O}_{\tilde{p}_0} \\ g \cdot p_0 &\mapsto g \cdot \tilde{p}_0. \end{aligned}$$

Proposition 5.6. *The natural projection $\pi: \mathcal{O}_{p_0} \rightarrow \mathcal{O}_{\tilde{p}_0}$ described above is a strong symplectic fibration.*

Proof. It is well known that π is a T -equivariant fiber bundle with symplectic fibers, isomorphic to $P_{\tilde{p}_0}/P_{p_0}$. Moreover, we can choose the complex structures J and \tilde{J} on \mathcal{O}_{p_0} and $\mathcal{O}_{\tilde{p}_0}$ so that π intertwines them. Hence, the claim is a direct consequence of the discussion in Example 4.2 (i). \square

In the case of coadjoint orbits we can explicitly lift ascending paths in $(\tilde{V}, \tilde{E}_{\text{GKM}})$.

Lemma 5.7. *Let $\varphi: \mathcal{O}_{\tilde{p}_0} \rightarrow \mathbb{R}$ be a generic component of the moment map. Given $p \in V$ and an ascending path $\tilde{\gamma}$ in $(\tilde{V}, \tilde{E}_{\text{GKM}})$ that begins at $\pi(p)$, there exists a unique path γ of length $|\tilde{\gamma}|$ in (V, E_{GKM}) such that*

- γ begins at p ,
- $V(\pi(\gamma)) = V(\tilde{\gamma})$, and
- $\lambda(\gamma_{i+1}) > \lambda(\gamma_i)$ for all i .

If $\tilde{\gamma}_{i+1} = s_{\beta_i}(\tilde{\gamma}_i)$ for $\beta_i \in R$ for each $1 \leq i \leq |\tilde{\gamma}|$, then the endpoint of γ is $w(p)$, where

$$w = s_{\beta_{|\tilde{\gamma}|}} s_{\beta_{|\tilde{\gamma}|-1}} \cdots s_{\beta_1}.$$

Proof. Fix $p \in V$. Since π is an equivariant fiber bundle, there is a unique **lift** γ of each path $\tilde{\gamma}$ starting at p , i.e., a unique path γ of length $|\tilde{\gamma}|$ in (V, E_{GKM}) that starts at p such that $\pi(\gamma_i, \gamma_{i+1}) = (\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ for all i . By Lemma 4.11 and Lemma 5.3, $\lambda(\gamma_{i+1}) > \lambda(\gamma_i)$ for all i exactly if $\tilde{\gamma}$ is ascending; this proves the first claim. The second claim is an consequence of the fact that $\pi: \mathcal{O}_{p_0} \rightarrow \mathcal{O}_{\tilde{p}_0}$ satisfies $\pi(w(p_0)) = w(\pi(p_0))$ for all $w \in W$. \square

5.1. Generic coadjoint orbits of type A_n . Let $G = SU(n+1)$, and let $T \subset G$ be the subtorus of diagonal matrices. We can identify the dual of the Lie algebra of T as $\mathfrak{t}^* = \{\mu \in (\mathbb{R}^{n+1})^* \mid \sum_{i=1}^{n+1} \mu_i = 0\}$; the roots are the vectors $x_i - x_j \in \mathfrak{t}^*$ for all $1 \leq i \neq j \leq n$. Here, and throughout this section, $\{x_i\}_{i=1}^{n+1}$ is the standard basis of $(\mathbb{R}^{n+1})^*$. Fix a point

$$\mu^j \in \mathfrak{t}^* \quad \text{such that } \mu_1^j < \cdots < \mu_j^j < \mu_{j+1}^j = \cdots = \mu_{n+1}^j \quad \text{for each } 0 \leq j \leq n;$$

for simplicity assume that $\mu_{j+1}^j = \mu_j^j + 1$. Let $(\mathcal{O}_{\mu^j}, \omega_j, \psi_j)$ be the coadjoint orbit through μ^j for each j . Observe that \mathcal{O}_{μ^0} is a single point and that \mathcal{O}_{μ^n} is isomorphic to $\mathcal{F}l(\mathbb{C}^{n+1})$, the variety of complete flags in \mathbb{C}^{n+1} . The stabilizer of μ^j is

$$P_{\mu^j} = S(U(1) \times \cdots \times U(1) \times U(n-j+1)) \quad \text{for all } j;$$

in particular, $P_{\mu^{j+1}} \subset P_{\mu^j}$. By Proposition 5.6, the natural projection map $p_j: \mathcal{O}_{\mu^{j+1}} \rightarrow \mathcal{O}_{\mu^j}$ is a strong symplectic fibration with fiber $P_{\mu^j}/P_{\mu^{j+1}} \simeq \mathbb{C}P^{n-j}$ for all $0 \leq j < n$.

Moreover, let $\varphi = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map that achieves its minimum value at μ^n . By Proposition 5.1, canonical classes $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{\mu^n}; \mathbb{Z})$ exist for all $p \in \mathcal{O}_{\mu^n}^T$; moreover $\Theta(r, r') = 1$ for each edge (r, r') in the associated canonical graph. Hence, Theorem 4.4 immediately implies that, for any $q \in \mathcal{O}_{\mu^n}^T$, we can express the restriction $\alpha_p(q)$ as a sum of terms $\Xi(\gamma)$ over paths $\gamma \in C(p, q)$, where each term is the product of distinct positive roots. In what follows we want to give an explicit description of the set $C(p, q)$ and of each term $\Xi(\gamma)$.

The Weyl group \mathcal{S}_{n+1} of G is the group of permutations of $n+1$ elements. Each element $\sigma \in \mathcal{S}_{n+1}$ can be represented in the one line notation by $\sigma = \sigma(1), \dots, \sigma(n+1)$;

the action of σ on a point $\mu = \sum_{i=1}^{n+1} \mu_i x_i \in \mathfrak{t}^*$ is given by $\sigma(\mu) = \sum_{i=1}^{n+1} \mu_i x_{\sigma(i)}$. Let $\pi_j = p_j \circ p_{j+1} \circ \cdots \circ p_{n-1}: \mathcal{O}_{\mu^n} \rightarrow \mathcal{O}_{\mu^j}$, and define

$$h(\sigma, \sigma') = \min\{j \in \{1, \dots, n\} \mid \pi_j(\sigma(\mu^n)) \neq \pi_j(\sigma'(\mu^n))\} \quad \text{for all } \sigma \neq \sigma' \text{ in } \mathcal{S}_{n+1}.$$

Fix any distinct σ and σ' in \mathcal{S}_{n+1} . Since $\pi_j(\sigma(\mu^n)) = \sigma(\mu^j)$ and $\mu_i^j = \mu_{j+1}^j$ exactly if $i > j$, $\pi_j(\sigma(\mu^n)) = \pi_j(\sigma'(\mu^n))$ exactly if $\sigma(i) = \sigma'(i)$ for all $0 \leq i \leq j$. Hence,

$$h(\sigma, \sigma') = \min\{j \in \{1, \dots, n\} \mid \sigma(j) \neq \sigma'(j)\}; \quad \text{therefore}$$

$$h(\sigma, \sigma s_{x_h - x_k}) = h \quad \text{for all } 1 \leq h < k \leq n+1.$$

Let $\bar{\psi}_j = \pi^*(\psi_j): \mathcal{O}_{\mu^n} \rightarrow \mathfrak{t}^*$ for all j . Since $\psi_j: \mathcal{O}_{\mu^j}^T \rightarrow \mathfrak{t}^*$ is the inclusion map, $\bar{\psi}_j(\sigma(\mu^n)) = \sum_{i=1}^{n+1} \mu_i^j x_{\sigma(i)}$ for all j . Since $\sum_{m=1}^{n+1} (x_{\sigma'(m)} - x_{\sigma(m)}) = 0$ and $\mu_j^j + 1 = \mu_{j+1}^j = \dots = \mu_{n+1}^j$, this implies that

$$\begin{aligned} \bar{\psi}_j(\sigma'(\mu^n)) - \bar{\psi}_j(\sigma(\mu^n)) &= \sum_{m=1}^{n+1} \mu_m^j (x_{\sigma'(m)} - x_{\sigma(m)}) = \sum_{m=1}^{n+1} (\mu_m^j - \mu_{j+1}^j) (x_{\sigma'(m)} - x_{\sigma(m)}) \\ &= \sum_{m=1}^j (\mu_m^j - \mu_{j+1}^j) (x_{\sigma'(m)} - x_{\sigma(m)}) \quad \text{for all } j, \text{ and so} \end{aligned}$$

$$\bar{\psi}_j(\sigma'(\mu^n)) - \bar{\psi}_j(\sigma(\mu^n)) = x_{\sigma(j)} - x_{\sigma'(j)} \quad \text{for all } j \leq h(\sigma, \sigma'); \quad \text{therefore,}$$

$$\bar{\psi}_h(\sigma s_{x_h - x_k}(\mu^n)) - \bar{\psi}_h(\sigma(\mu^n)) = x_{\sigma(h)} - x_{\sigma(k)} = \sigma(x_h - x_k) \quad \text{for all } 1 \leq h < k \leq n+1.$$

Therefore, the next proposition follows directly from Theorem 4.4 and Proposition 5.1. (Here, we use the fact that $h(\sigma_i, \sigma_{i+1}) \leq h(\sigma_i, \sigma_{|\underline{\sigma}|+1})$ for any $\underline{\sigma} = (\sigma_1, \dots, \sigma_{|\underline{\sigma}|+1}) \in C(p, q)$.)

Proposition 5.8. *Let $(\mathcal{O}_{\mu^n}, \omega_n, \psi_n)$ be a generic coadjoint orbit of $SU(n+1)$. Let $\varphi_n = \psi_n^\xi$ be a generic component of the moment map that achieves its minimum at μ^n ; assume that $\{x_1 - x_2, \dots, x_{n-1} - x_n\}$ is the associated set simple roots. Let \mathcal{S}_{n+1} be the Weyl group of $SU(n+1)$ and define*

$$E = \{(\sigma, \sigma s_\beta) \in \mathcal{S}_{n+1} \times \mathcal{S}_{n+1} \mid l(\sigma s_\beta) = l(\sigma) + 1 \text{ and } \beta \text{ is a root}\}; \text{ moreover}$$

$$h(\sigma, \sigma s_{x_h - x_k}) = h \quad \text{for all } \sigma \in \mathcal{S}_{n+1} \text{ and } 1 \leq h < k \leq n+1.$$

- (1) *Given p and q in $\mathcal{O}_{\mu^n}^T$, let $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{\mu^n}; \mathbb{Z})$ be the canonical class at p , and let $\Sigma(p, q)$ denote the set of paths $\underline{\sigma} = (\sigma_1, \dots, \sigma_{|\underline{\sigma}|+1})$ in (\mathcal{S}_{n+1}, E) such that $\sigma_1(\mu^n) = p$ and $\sigma_{|\underline{\sigma}|+1}(\mu^n) = q$. Then*

$$\alpha_p(q) = \sum_{\underline{\sigma} \in C(p, q)} \Xi(\underline{\sigma}), \quad \text{where}$$

$$\Xi(\underline{\sigma}) = \Lambda_q^- \prod_{i=1}^{|\underline{\sigma}|} \frac{1}{x_{\sigma_i(h_i)} - x_{\sigma_{|\underline{\sigma}|+1}(h_i)}} \quad \text{for all } \underline{\sigma} \in C(p, q), \text{ and}$$

$$C(p, q) = \{\underline{\sigma} = (\sigma_1, \dots, \sigma_{|\underline{\sigma}|+1}) \in \Sigma(p, q) \mid h(\sigma_1, \sigma_2) \leq h(\sigma_2, \sigma_3) \leq \dots \leq h(\sigma_{|\underline{\sigma}|}, \sigma_{|\underline{\sigma}|+1})\} \text{ and}$$

$$h_i = h(\sigma_i, \sigma_{i+1}) \quad \text{for all } i.$$

(2) For each path $\underline{\sigma} \in C(p, q)$, $\Xi(\underline{\sigma})$ is the product of distinct positive roots.

5.2. Generic coadjoint orbits of type C_n . Let $G = Sp(n)$ be the symplectic group, i.e. the quaternionic unitary group $U(n; \mathbb{H})$, and let $T \subset G$ be a maximal torus. We can identify the dual of the Lie algebra of T as $\mathfrak{t}^* = (\mathbb{R}^n)^*$; the roots are the vectors $\pm x_i \pm x_j$ and $\pm 2x_i$ for all $1 \leq i \neq j \leq n$. Fix a point

$$\mu^j \in \mathfrak{t}^* \quad \text{such that } \mu_1^j < \cdots < \mu_j^j < 0 = \mu_{j+1}^j = \cdots = \mu_n^j \quad \text{for each } 0 \leq j \leq n;$$

for simplicity assume that $\mu_j^j = -1$. Let $(\mathcal{O}_{\mu^j}, \omega_j, \psi_j)$ be the coadjoint orbit through μ^j for each j . The stabilizer of μ^j is

$$P_{\mu^j} = S^1 \times \cdots \times S^1 \times U(n-j; \mathbb{H}) \quad \text{for all } j;$$

in particular, $P_{\mu^{j+1}} \subset P_{\mu^j}$. By Proposition 5.6, the natural projection map $p_j: \mathcal{O}_{\mu^{j+1}} \rightarrow \mathcal{O}_{\mu^j}$ is a strong symplectic fibration with fiber $P_{\mu^j}/P_{\mu^{j+1}} \simeq \mathbb{CP}^{2(n-j)-1}$ for all $0 \leq j < n$.

Moreover, let $\varphi = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map that achieves its minimum value at μ^n . By Proposition 5.1, canonical classes $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{\mu^n}; \mathbb{Z})$ exist for all $p \in \mathcal{O}_{\mu^n}^T$; moreover, $\Theta(r, r') = 1$ for each edge (r, r') in the associated canonical graph. Hence, Theorem 4.4 immediately implies that, for any $q \in \mathcal{O}_{\mu^n}^T$, we can express the restriction $\alpha_p(q)$ as a sum of terms $\Xi(\gamma)$ over paths $\gamma \in C(p, q)$, where each term is a product of distinct positive roots. In what follows we want to give an explicit description of the set $C(p, q)$ and the terms $\Xi(\gamma)$.

The Weyl group W of G is the group of signed permutations of n elements. Each element $\tau \in W$ can be represented in one line notation by $\tau = (-1)^{\epsilon_1} \sigma(1), \dots, (-1)^{\epsilon_n} \sigma(n)$, where $\epsilon_i \in \{0, 1\}$ for all i and $\sigma \in \mathcal{S}_n$; the action of τ on a point $\mu = \sum_{i=1}^n \mu_i x_i \in \mathfrak{t}^*$ is given by $\tau(\mu) = \sum_{i=1}^n (-1)^{\epsilon_i} \mu_i x_{\sigma(i)}$. Let $\pi_j = p_j \circ p_{j+1} \circ \cdots \circ p_{n-1}: \mathcal{O}_{\mu^n} \rightarrow \mathcal{O}_{\mu^j}$, and define

$$h(\tau, \tau') = \min\{j \in \{1, \dots, n\} \mid \pi_j(\tau(\mu^n)) \neq \pi_j(\tau'(\mu^n))\} \quad \text{for all } \tau \neq \tau' \text{ in } W.$$

Fix any distinct $\tau = (-1)^{\epsilon_1} \sigma(1), \dots, (-1)^{\epsilon_n} \sigma(n)$ and $\tau' = (-1)^{\epsilon'_1} \sigma'(1), \dots, (-1)^{\epsilon'_n} \sigma'(n)$ in W . Since $\pi_j(\tau(\mu^n)) = \tau(\mu^j)$ and $\mu_i^j = \mu_{j+1}^j = 0$ exactly if $i > j$, $\pi_j(\tau(\mu^n)) = \pi_j(\tau'(\mu^n))$ exactly if $\sigma(i) = \sigma'(i)$ and $\epsilon_i = \epsilon'_i$ for all $0 \leq i \leq j$. Hence,

$$h(\tau, \tau') = \min\{j \in \{1, \dots, n\} \mid \sigma(j) \neq \sigma'(j) \text{ or } \epsilon_j \neq \epsilon'_j\}; \quad \text{therefore,}$$

$$h(\tau, \tau s_{x_h \pm x_k}) = h \quad \text{for all } 1 \leq h < k \leq n, \quad \text{and} \quad h(\tau, \tau s_{2x_h}) = h \quad \text{for all } 1 \leq h \leq n.$$

Let $\bar{\psi}_j = \pi^*(\psi_j): \mathcal{O}_{\mu^n} \rightarrow \mathfrak{t}^*$ for all j . Since $\psi_j: \mathcal{O}_{\mu^j} \rightarrow \mathfrak{t}^*$ is the inclusion map, $\bar{\psi}_j(\tau(\mu^n)) = \sum_{i=1}^n \mu_i^j (-1)^{\epsilon_i} x_{\sigma(i)}$ for all j . Hence,

$$\bar{\psi}_j(\tau'(\mu^n)) - \bar{\psi}_j(\tau(\mu^n)) = \sum_{m=1}^j \mu_m^j \left((-1)^{\epsilon'_m} x_{\sigma'(m)} - (-1)^{\epsilon_m} x_{\sigma(m)} \right) \quad \text{for all } j, \text{ and so}$$

$$\bar{\psi}_j(\tau'(\mu^n)) - \bar{\psi}_j(\tau(\mu^n)) = \left((-1)^{\epsilon_j} x_{\sigma(j)} - (-1)^{\epsilon'_j} x_{\sigma'(j)} \right) \quad \text{for all } j \leq h(\tau, \tau'); \quad \text{therefore,}$$

$$\bar{\psi}_h(\tau s_{x_h \pm x_k} \mu^n) - \bar{\psi}_h(\tau(\mu^n)) = (-1)^{\epsilon_h} x_{\sigma(h)} \mp (-1)^{\epsilon_k} x_{\sigma(k)} = \sigma(x_h \pm x_k)$$

for all $1 \leq h < k \leq n$, and

$$\bar{\psi}_h(\tau s_{2x_h} \mu^n) - \bar{\psi}_h(\tau(\mu^n)) = (-1)^{\epsilon_h} (2x_{\sigma(h)}) = \sigma(2x_h) \quad \text{for all } 1 \leq h \leq n.$$

Therefore, the next proposition follows directly from Theorem 4.4 and Proposition 5.1. (Here, we use the fact that $h(\tau_i, \tau_{i+1}) \leq h(\tau_i, \tau_{|\underline{\tau}|+1})$ for any $\underline{\tau} = (\tau_1, \dots, \tau_{|\underline{\tau}|+1}) \in C(p, q)$.)

Proposition 5.9. *Let $(\mathcal{O}_{\mu^n}, \omega_n, \psi_n)$ be a generic coadjoint orbit of $Sp(n)$. Let $\varphi_n = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map that achieves its minimum at μ^n ; assume that $\{x_1 - x_2, \dots, x_{n-1} - x_n, 2x_n\}$ is the associated set of simple roots. Let W be the Weyl group of $Sp(n)$ and define*

$$E = \{(\tau, \tau s_\beta) \in W \times W \mid l(\tau s_\beta) = l(\tau) + 1 \text{ and } \beta \text{ is a root}\}; \text{ moreover}$$

$$h(\tau, \tau s_{x_h \pm x_k}) = h \quad \text{for all } \tau \in W \text{ and } 1 \leq h < k \leq n \text{ and}$$

$$h(\tau, \tau s_{2x_h}) = h \quad \text{for all } \tau \in W \text{ and } 1 \leq h \leq n.$$

- (1) *Given p and q in $\mathcal{O}_{\mu^n}^T$, let $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{\mu^n}; \mathbb{Z})$ be the canonical class at p , and let $\Sigma(p, q)$ denote the set of paths $\underline{\tau} = (\tau_1, \dots, \tau_{|\underline{\tau}|+1})$ in (W, E) such that $\tau_1(\mu^n) = p$ and $\tau_{|\underline{\tau}|+1}(\mu^n) = q$. Let $\tau_i = (-1)^{\epsilon_i^i} \sigma_i(1), \dots, (-1)^{\epsilon_i^n} \sigma_i(n)$ for all i . Then*

$$\alpha_p(q) = \sum_{\underline{\tau} \in C(p, q)} \Xi(\underline{\tau}), \quad \text{where}$$

$$\Xi(\underline{\tau}) = \Lambda_q^- \prod_{i=1}^{|\underline{\tau}|} \frac{1}{(-1)^{\epsilon_{h_i}^{|\underline{\tau}|+1}} x_{\sigma_{|\underline{\tau}|+1}(h_i)} - (-1)^{\epsilon_{h_i}^i} x_{\sigma_i(h_i)}} \quad \text{for all } \underline{\tau} \in C(p, q), \text{ with}$$

$$C(p, q) = \{\underline{\tau} = (\tau_1, \dots, \tau_{|\underline{\tau}|+1}) \in \Sigma(p, q) \mid h(\tau_1, \tau_2) \leq h(\tau_2, \tau_3) \leq \dots \leq h(\tau_{|\underline{\tau}|}, \tau_{|\underline{\tau}|+1})\} \quad \text{and}$$

$$h_i = h(\tau_i, \tau_{i+1}) \quad \text{for all } i.$$

- (2) *For each path $\underline{\tau} \in C(p, q)$, $\Xi(\underline{\tau})$ is the product of distinct positive roots.*

5.3. Generic coadjoint orbits of type B_n . Let $G = SO(2n+1)$ and let $T \subset G$ be a maximal torus. We can identify the dual of the Lie algebra of T as $\mathfrak{t}^* = (\mathbb{R}^n)^*$; the roots are the vectors $\pm x_i \pm x_j \in \mathfrak{t}^*$ and $\pm x_i \in \mathfrak{t}^*$ for all $1 \leq i \neq j \leq n$. Fix a point

$$\mu^j \in \mathfrak{t}^* \quad \text{such that } \mu_1^j < \dots < \mu_j^j < 0 = \mu_{j+1}^j = \dots = \mu_n^j \quad \text{for each } 0 \leq j \leq n;$$

for simplicity assume that $\mu_j^j = -1$. Let $(\mathcal{O}_{\mu^j}, \omega_j, \psi_j)$ be the coadjoint orbit through μ^j for each j . The stabilizer of μ^j is

$$SO(2) \times \dots \times SO(2) \times SO(2n - 2j + 1) \quad \text{for all } j;$$

in particular, $P_{\mu^{j+1}} \subset P_{\mu^j}$. By Proposition 5.6, the natural projection map $p_j: \mathcal{O}_{\mu^{j+1}} \rightarrow \mathcal{O}_{\mu^j}$ is a strong symplectic fibration with fiber $P_{\mu^j}/P_{\mu^{j+1}} \simeq Gr_2^+(\mathbb{R}^{2n-2j+1})$ for all $0 \leq j < n$. Here, $Gr_2^+(\mathbb{R}^k)$ denotes the Grassmannian of oriented two planes in \mathbb{R}^k .

Moreover, let $\varphi = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map that achieves its minimum value at μ^n . By Proposition 5.1, canonical classes $\alpha_p \in H_T^{2\lambda(p)}(\mathcal{O}_{\mu^n}; \mathbb{Z})$ exist for all $p \in \mathcal{O}_{\mu^n}^T$; moreover $\Theta(r, r') = 1$ for each edge (r, r') in the associated canonical graph. Hence, since $H^*(Gr_2^+(\mathbb{R}^{2n-2j+1}); \mathbb{Z}[\frac{1}{2}]) \simeq H^*(\mathbb{CP}^{2n-2j-1}; \mathbb{Z}[\frac{1}{2}])$, Theorem 4.4 immediately implies that, for any $q \in \mathcal{O}_{\mu^n}^T$, we can express the restriction $\alpha_p(q)$ as a sum of terms $\Xi(\gamma)$ over paths $\gamma \in C(p, q)$, where each term is a polynomial in the simple roots with positive *rational* coefficients; more precisely, $\Xi(\gamma)$ is the product of distinct positive roots and a constant that is a (possibly negative) power of 2.

The main result of this section is an inductive *positive integral formula* that expresses the restriction $\alpha_p(q)$ in terms of products of distinct positive roots with positive integer coefficients, and the restriction of canonical classes on a generic coadjoint orbit of type

B_{n-1} for all $p, q \in \mathcal{O}_{\mu^n}^T$. To find this formula, we will apply Corollary 4.7 to the natural projection $\pi: \mathcal{O}_{\mu^n} \rightarrow \mathcal{O}_{\mu^1}$.

Given p and $s \in \mathcal{O}_{\mu^n}^T$, let $\Sigma(p, s)$ be the set of paths in the canonical graph (V, E) associated to \mathcal{O}_{μ^n} , and let $\overline{\Sigma}(p, s) \subset \Sigma(p, s)$ be the subset of horizontal paths; see Definition 4.8. Given any $\gamma \in \overline{\Sigma}(p, s)$, by Lemmas 4.10 and 4.11, its projection $\tilde{\gamma} = \pi(\gamma)$ is an ascending path in the GKM graph $(\tilde{V}, \tilde{E}_{\text{GKM}})$ associated to \mathcal{O}_{μ^1} . Note that $(\tilde{V}, \tilde{E}_{\text{GKM}})$ is a complete graph; in ascending order, the T -fixed points of \mathcal{O}_{μ^1} are

$$-x_1, -x_2, \dots, -x_n, x_n, \dots, x_2, x_1.$$

Finally, given a path $\tilde{\gamma}$ in $(\tilde{V}, \tilde{E}_{\text{GKM}})$, let $V(\tilde{\gamma})$ be the set of vertices of $\tilde{\gamma}$

$$V(\tilde{\gamma}) = \{\tilde{\gamma}_1, \dots, \tilde{\gamma}_{|\gamma|+1}\} \subseteq \mathcal{O}_{\mu^1}^T.$$

To state our main theorem, we will need the following definitions.

Definition 5.10. A path $\gamma \in \overline{\Sigma}(p, s)$ with $\pi(\gamma) = \tilde{\gamma}$ is **incomplete** if both the following conditions are satisfied:

- (i) $\{-\pi(s), \pi(s)\} \subset V(\tilde{\gamma})$, and
- (ii) $\tilde{\gamma}$ does not contain the edge $(-x_j, x_j)$ for any $j = 1, \dots, n$.

Otherwise γ is **complete**.

Definition 5.11. A path $\gamma \in \overline{\Sigma}(p, s)$ with $\pi(\gamma) = \tilde{\gamma}$ is **relevant** if either it is complete or if it is incomplete and $x_{k(\gamma)+1} \in V(\tilde{\gamma})$, where $k(\gamma) = \max\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\}$. (Observe that condition (i) in the definition above implies that if γ is incomplete then $\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\} \neq \emptyset$ and condition (ii) implies that $k(\gamma) < n$.)

For every p and $s \in \mathcal{O}_{\mu^n}^T$ and every path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ in $\overline{\Sigma}(p, s)$ define

$$P(\gamma) = \tilde{\Lambda}_{\pi(s)}^- \prod_{i=1}^{|\gamma|} \frac{1}{\pi(s) - \pi(\gamma_i)} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\eta(\gamma_i, \gamma_{i+1})}.$$

Observe that since $\tilde{\psi}: \mathcal{O}_{\mu^1}^T \rightarrow \mathfrak{t}^*$ is the inclusion, by Theorem 1.10 and Proposition 5.4 this is precisely the term defined in Corollary 4.7.

The main theorem of this section can be stated as follows.

Theorem 5.12. Let $\pi: (\mathcal{O}_{\mu^n}, \omega_n, \psi_n) \rightarrow (\mathcal{O}_{\mu^1}, \omega_1, \psi_1)$ be the natural projection map. Let $\varphi = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map. Consider the canonical classes $\alpha_p \in H_T^*(\mathcal{O}_{\mu^n}; \mathbb{Z})$ for all $p \in \mathcal{O}_{\mu^n}^T$. For all $s \in \widehat{M}_q^T$ let $\hat{\alpha}_s \in H_T^*(\widehat{M}_q; \mathbb{Z})$ be the canonical class on the fiber $\widehat{M}_q = \pi^{-1}(\pi(q))$.

- (1) Given $s \in \widehat{M}_q^T$, let $R(p, s) \subset \overline{\Sigma}(p, s)$ denote the set of relevant paths from p to s . Then

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in R(p, s)} Q(\gamma) \right) \hat{\alpha}_s(q), \quad \text{where for every } \gamma \in R(p, s)$$

$$Q(\gamma) = \begin{cases} P(\gamma) & \text{if } \gamma \text{ is complete} \\ P(\gamma) \frac{2\pi(s)}{\pi(s) + x_{k(\gamma)+1}} & \text{if } \gamma \text{ is incomplete.} \end{cases}$$

- (2) $Q(\gamma)$ is the product of distinct positive roots and a constant which is either 1 or 2.

Before proving Theorem 5.12 we need to analyze how the expression of $P(\gamma)$ is related to whether γ is complete or incomplete.

Proposition 5.13. *Let $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ be a path in $\overline{\Sigma}(p, s)$. Let $\tilde{\gamma} = \pi(\gamma)$ and let $SV(\tilde{\gamma})$ be the skipped vertices of $\tilde{\gamma}$; see Definition 4.22. Then*

$$P(\gamma) = c \prod_{r \in SV(\tilde{\gamma})} \eta(r, \pi(s)), \quad \text{where } c = \begin{cases} 1 \text{ or } 2 & \text{if } \gamma \text{ is complete, and} \\ \frac{1}{2} & \text{if } \gamma \text{ is incomplete.} \end{cases}$$

Proof. Let $m(r, r')$ be the magnitude of an edge $(r, r') \in \tilde{E}_{\text{GKM}}$ with respect to $\tilde{\psi}$. (See Definition 4.16.) The edge $(-x_j, x_j)$ has magnitude 2 for all j ; all the other edges have magnitude 1. Since $\tilde{\gamma}$ is ascending, it can have at most one edge of type $(-x_j, x_j)$. Therefore, the claim follows from Lemma 4.23, Proposition 5.4, and the definition of complete and incomplete path. \square

We also need the following two lemmas.

Lemma 5.14. *Let γ be a path in $\overline{\Sigma}(p, s)$. If $\{-x_l, x_l\} \subset V(\pi(\gamma))$ for some $l < n$, then $\{-x_{l+1}, x_{l+1}\} \cap V(\pi(\gamma)) \neq \emptyset$.*

Lemma 5.15. *Let γ be a path in $\overline{\Sigma}(p, s)$ such that $\{-x_l, x_l\} \subset V(\pi(\gamma))$ for some l . If $\{-x_{l+1}, x_{l+1}\} \cap V(\pi(\gamma)) = \{x_{l+1}\}$ for some l , then there exists a unique path $\gamma' \in \overline{\Sigma}(p, s)$ such that $V(\pi(\gamma'))$ is obtained from $V(\pi(\gamma))$ by replacing the vertex x_{l+1} with $-x_{l+1}$. That is, $-x_{l+1} \in V(\pi(\gamma'))$ and $V(\pi(\gamma')) \setminus \{-x_{l+1}\} = V(\pi(\gamma)) \setminus \{x_{l+1}\}$. A similar claim holds if $\{-x_{l+1}, x_{l+1}\} \cap V(\pi(\gamma)) = \{-x_{l+1}\}$.*

To simplify the proof of these lemmas, let $s_l = s_{x_l - x_{l+1}}$ denote the reflection across the root $x_l - x_{l+1}$ for all $l \in \{1, \dots, n-1\}$,

We recall the following relations; for all $l \in \{1, \dots, n-1\}$ and $j \in \{1, 2, \dots, n\}$ with $j \notin \{l, l+1\}$

$$(22) \quad \begin{aligned} s_{x_l} &= s_l s_{x_{l+1}} s_l \\ s_{x_l \pm x_j} &= s_l s_{x_{l+1} \pm x_j} s_l \\ s_{x_l + x_{l+1}} &= s_l s_{x_l + x_{l+1}} s_l \end{aligned}$$

Proof of Lemma 5.14. Let $\tilde{\gamma} = \pi(\gamma)$. Suppose that, on the contrary, $\{-x_l, x_l\} \subset V(\tilde{\gamma})$ but $\{-x_{l+1}, x_{l+1}\} \cap V(\tilde{\gamma}) = \emptyset$. Let $\tilde{\gamma}'$ be the ascending path in $(\tilde{V}, \tilde{E}_{\text{GKM}})$ such that $V(\tilde{\gamma}') = \{-x_{l+1}, x_{l+1}\} \cup V(\tilde{\gamma})$. There exists $\beta_i \in R$ such that $\tilde{\gamma}_{i+1} = s_{\beta_i}(\tilde{\gamma}_i)$ for all $i = 1, \dots, |\tilde{\gamma}|$, and there exists $\delta_i \in R$ such that $\tilde{\gamma}'_{i+1} = s_{\delta_i}(\tilde{\gamma}'_i)$ for all $i = 1, \dots, |\tilde{\gamma}'|$. Define w and w' in the Weyl group W of G by

$$w = s_{\beta_{|\tilde{\gamma}|}} s_{\beta_{|\tilde{\gamma}|-1}} \cdots s_{\beta_1} \quad \text{and} \quad w' = s_{\delta_{|\tilde{\gamma}'|}} s_{\delta_{|\tilde{\gamma}'|-1}} \cdots s_{\delta_1}.$$

- If $\tilde{\gamma} = (\dots, -x_l, x_l, \dots)$, then $w = w_1 s_{x_l} w_2$ and $w' = w_1 s_l s_{x_{l+1}} s_l w_2$ for some w_1 and $w_2 \in W$. Hence $w = w'$ by (22).
- If $(-x_l, x_l)$ is not an edge of $\tilde{\gamma}$, then there exists i and $h > l$ so that $w = w_1 s_{x_l \pm x_h} w_0 s_{x_l \pm x_i} w_2$ and $w' = w_1 s_l s_{x_{l+1} \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l w_2$ for some w_0, w_1 , and $w_2 \in W$ such that w_0 commutes with s_l . Hence by (22) we again have $w' = w_1 s_l s_{x_{l+1} \pm x_h} s_l w_0 s_l s_{x_{l+1} \pm x_i} s_l w_2 = w_1 s_{x_l \pm x_h} w_0 s_{x_l \pm x_i} w_2 = w$.

Moreover, by Lemma 5.7 there exists a path γ' of length $|\tilde{\gamma}'|$ in (V, E_{GKM}) that starts at p such that $V(\pi(\gamma')) = V(\tilde{\gamma}')$ and $\lambda(\gamma'_{i+1}) > \lambda(\gamma'_i)$ for all $1 \leq i \leq |\tilde{\gamma}'|$. Moreover, since $w = w'$, the endpoints $s = w(p)$ of γ and $s' = w'(p)$ of γ' are equal. On the other hand, the

fact that $\gamma \in \overline{\Sigma}(p, s) \subset \Sigma(p, s)$ implies that $\lambda(s) - \lambda(p) = |\tilde{\gamma}|$. Moreover, $\lambda(\gamma'_{i+1}) > \lambda(\gamma'_i)$ for all $1 \leq i \leq |\tilde{\gamma}'|$. Hence, $\lambda(s') - \lambda(p) \geq |\tilde{\gamma}'| = |\tilde{\gamma}| + 2$. Since $s = s'$, this is impossible. \square

Proof of Lemma 5.15. Let $\tilde{\gamma} = \pi(\gamma)$. Assume that $\{-x_{l+1}, x_{l+1}\} \cap V(\tilde{\gamma}) = \{x_{l+1}\}$. Let $\tilde{\gamma}'$ be the ascending path in $(\tilde{V}, \tilde{E}_{\text{GKM}})$ such that $V(\tilde{\gamma}')$ is obtained from $V(\tilde{\gamma})$ by replacing the vertex x_{l+1} with $-x_{l+1}$. As before, there exists $\beta_i \in R$ such that $\gamma_{i+1} = s_{\beta_i}(\gamma_i)$ for all $i = 1, \dots, |\gamma|$, and there exists $\delta_i \in R$ such that $\gamma'_{i+1} = s_{\delta_i}(\gamma'_i)$ for all $i = 1, \dots, |\gamma'|$. By Lemma 5.7, this implies that $\tilde{\gamma}_{i+1} = s_{\beta_i}(\tilde{\gamma}_i)$ and $\tilde{\gamma}'_{i+1} = s_{\delta_i}(\tilde{\gamma}'_i)$ for all i . Define w and $w' \in W$ by

$$w = s_{\beta_{|\tilde{\gamma}|}} s_{\beta_{|\tilde{\gamma}|-1}} \cdots s_{\beta_1} \quad \text{and} \quad w' = s_{\delta_{|\tilde{\gamma}'|}} s_{\delta_{|\tilde{\gamma}'|-1}} \cdots s_{\delta_1}.$$

- If $\tilde{\gamma} = (\dots, -x_l, x_{l+1}, x_l, \dots)$ then $w = w_1 s_l s_{x_l+x_{l+1}} w_2$ and $w' = w_1 s_{x_l+x_{l+1}} s_l w_2$ for some $w_1, w_2 \in W$. Hence $w = w'$ by (22).
- If $(-x_1, x_{l+1})$ is not an edge of $\tilde{\gamma}$, then there exists h and $i > l+1$ so that $w = w_1 s_l s_{x_{l+1} \pm x_h} w_0 s_{x_l \pm x_i} w_2$ and $w' = w_1 s_{x_l \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l w_2$ for some w_0, w_1 , and $w_2 \in W$ such that w_0 commutes with s_l . Hence again by (22) we have $w = w_1 s_l s_{x_{l+1} \pm x_h} s_l w_0 s_{x_l \pm x_i} w_2 = w_1 s_{x_l \pm x_h} w_0 s_{x_{l+1} \pm x_i} s_l w_2 = w'$.

One the other hand, the fact that $\gamma \in \overline{\Sigma}(p, s)$ implies that $\lambda(s) - \lambda(p) = |\gamma|$. Moreover, since $\tilde{\gamma}'$ is an ascending path, Lemma 4.11 and Lemma 5.3 together imply that $\lambda(\gamma'_{i+1}) - \lambda(\gamma'_i) \geq 1$ for all $1 \leq i \leq |\gamma'| = |\gamma|$. But this is impossible unless $\lambda(\gamma'_{i+1}) - \lambda(\gamma'_i) = 1$ for all i , which implies that $\gamma' \in \overline{\Sigma}(p, s)$. \square

We are now ready to prove Theorem 5.12

Proof of Theorem 5.12. Since Proposition 5.6 implies that π is a strong symplectic fibration and ψ_1 is the inclusion, Corollary 4.7 and Proposition 5.1 together imply that

$$\alpha_p(q) = \sum_{s \in \tilde{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p, s)} P(\gamma) \right) \hat{\alpha}_s(q).$$

If $\gamma \in \overline{\Sigma}(p, s)$ is complete, then γ is relevant and $P(\gamma) = Q(\gamma)$. On the other hand, by Lemmas 5.14 and 5.15, the set of incomplete paths can be decomposed into pairs of paths γ and γ' , so that $V(\tilde{\gamma}') = V(\pi(\gamma'))$ is obtained from $V(\tilde{\gamma}) = V(\pi(\gamma))$ by replacing $x_{k(\gamma)+1}$ by $-x_{k(\gamma)+1}$, where $k(\gamma) = \max\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\}$. Observe that by definition of γ and γ' we have $SV(\tilde{\gamma}) \setminus \{-x_{k(\gamma)+1}\} = SV(\tilde{\gamma}') \setminus \{x_{k(\gamma)+1}\}$. Additionally, by the definition of $k(\gamma)$, $\pi(s) \neq \pm x_{k(\gamma)+1}$, and so $\eta(\pm x_{k(\gamma)+1}, \pi(s)) = \pi(s) \mp x_{k(\gamma)+1}$. Hence by Proposition 5.13

$$\begin{aligned} P(\gamma) + P(\gamma') &= \frac{1}{2} \left(\prod_{r \in SV(\tilde{\gamma}) \cap SV(\tilde{\gamma}')} \eta(r, \pi(s)) \right) (\eta(-x_{k(\gamma)+1}, \pi(s)) + \eta(x_{k(\gamma)+1}, \pi(s))) \\ &= \pi(s) \prod_{r \in SV(\tilde{\gamma}) \cap SV(\tilde{\gamma}')} \eta(r, \pi(s)) = Q(\gamma). \end{aligned}$$

Since γ is relevant, but γ' is not, this implies that

$$\sum_{\gamma \in \overline{\Sigma}(p, s)} P(\gamma) = \sum_{\gamma \in R(p, s)} Q(\gamma).$$

This proves part (1) of Theorem 5.12. Finally, by the definition of $SV(\tilde{\gamma})$, $\eta(r, \pi(s))$ is a positive root for all $r \in SV(\tilde{\gamma})$. Hence if γ is complete then $Q(\gamma) = P(\gamma)$ is the product

of distinct positive roots and a constant which is either 1 or 2. Moreover, by definition of incomplete path, $\pi(s)$ must be a positive root. Hence, if γ is incomplete then $Q(\gamma)$ is also the product of distinct positive roots. \square

Example 5.16 Let \mathcal{O}_{μ^2} be the coadjoint orbit of $SO(3)$ through $\mu^2 = -2x_1 - x_2$. The associated GKM graph (V, E_{GKM}) has eight vertices, $-2x_1 - x_2$, $-2x_1 + x_2$, $-x_1 - 2x_2$, $x_1 - 2x_2$, $-x_1 + 2x_2$, $x_1 + 2x_2$, $2x_1 - x_2$, $2x_1 + x_2$. Let $\varphi = \psi^\xi$ be the generic component of the moment map that achieves its minimum at μ^2 . Let \mathcal{O}_{μ^1} be the coadjoint orbit through $\mu^1 = -x_1$. The associated GKM graph $(\tilde{V}, \tilde{E}_{\text{GKM}})$ has four vertices, $-x_1$, $-x_2$, x_2 and x_1 . It's easy to see that $\pi((-1)^{\epsilon_1} 2x_{\sigma(1)} + (-1)^{\epsilon_2} x_{\sigma(2)}) = (-1)^{\epsilon_1} x_{\sigma(1)}$, for all $\sigma \in \mathcal{S}_2$ and $\epsilon_i \in \{0, 1\}$. Let $p = -2x_1 + x_2$ and $q = 2x_1 + x_2$. We want to compute $\alpha_p(q)$ using $\pi: \mathcal{O}_{\mu^2} \rightarrow \mathcal{O}_{\mu^1}$.

Since $\pi^{-1}(\pi(q))$ is composed by two points, $s = 2x_1 - x_2$ and q , we need to find the sets of paths $\bar{\Sigma}(p, s)$ and $\bar{\Sigma}(p, q)$, and the corresponding subsets of relevant paths $R(p, s)$ and $R(p, q)$. It's easy to see that

- $\bar{\Sigma}(p, s) = \{\gamma^1, \gamma^2\}$, where $\tilde{\gamma}^1 = \pi(\gamma^1) = (-x_1, x_2, x_1)$ and $\tilde{\gamma}^2 = (-x_1, -x_2, x_1)$; so the paths γ^1 and γ^2 are incomplete, and γ^1 is relevant. Hence $R(p, q) = \{\gamma^1\}$ and $Q(\gamma^1) = x_1$.
- $\bar{\Sigma}(p, q) = \{\gamma^3\}$, where $\tilde{\gamma}^3 = \pi(\gamma^3) = (-x_1, -x_2, x_2, x_1)$. Hence, since $\bar{\Sigma}(p, q)$ is composed by one path only, γ^3 must be complete, and hence relevant. Moreover $Q(\gamma^3) = 1$.

By Theorem 5.12, we have that

$$\alpha_p(q) = Q(\gamma^1)\hat{\alpha}_s(q) + Q(\gamma^3)\hat{\alpha}_q(q)$$

Since $\pi: \mathcal{O}_{\mu^2} \rightarrow \mathcal{O}_{\mu^1}$ is a \mathbb{CP}^1 -bundle, we have that $\hat{\alpha}_s(q) = 1$ and $\hat{\alpha}_q(q) = x_2$, and we can conclude that

$$\alpha_p(q) = x_1 + x_2.$$

5.4. Generic coadjoint orbit of type D_n . Let $G = SO(2n)$ and let $T \subset G$ be a maximal torus. We can identify the dual of the Lie algebra of T as $\mathfrak{t}^* = (\mathbb{R}^n)^*$; the roots are the vectors $\pm x_i \pm x_j \in \mathfrak{t}^*$ for all $1 \leq i \neq j \leq n$. Fix a point

$$\mu^j \in \mathfrak{t}^* \quad \text{such that } \mu_1^j < \cdots < \mu_j^j < 0 = \mu_{j+1}^j = \cdots = \mu_n^j \quad \text{for each } 0 \leq j \leq n;$$

for simplicity assume that $\mu_j^j = -1$. Let $(\mathcal{O}_{\mu^j}, \omega_j, \psi_j)$ be the coadjoint orbit through μ^j for each j . The stabilizer of μ^j is

$$SO(2) \times \cdots \times SO(2) \times SO(2n - 2j) \quad \text{for all } j;$$

in particular, $P_{\mu^{j+1}} \subset P_{\mu^j}$. By Proposition 5.6, the natural projection map $p_j: \mathcal{O}_{\mu^{j+1}} \rightarrow \mathcal{O}_{\mu^j}$ is a strong symplectic fibration with fiber $P_{\mu^j}/P_{\mu^{j+1}} \simeq Gr_2^+(\mathbb{R}^{2n-2j})$ for all $0 \leq j < n$. However, since $H^{2n-2j-2}(Gr_2^+(\mathbb{R}^{2n-2j}); \mathbb{R}) = \mathbb{R}^2$, we can not use Theorem 4.4 to express the restriction $\alpha_p(q)$ as a sum of polynomial terms.

Nevertheless, the main result of this section is an inductive *positive integral formula* that for all $n \geq 4$ expresses the restriction $\alpha_p(q)$ in terms of products of distinct roots, and the restriction of canonical classes on a generic coadjoint orbit of type D_{n-1} for all $p, q \in \mathcal{O}_{\mu^n}^T$. (If $n = 3$ then \mathcal{O}_{μ^n} is also the complete flag on \mathbb{C}^4 , and so we can use the techniques of § 5.1.) To find this formula we will proceed as in section 5.3, we will apply Corollary 4.7 to the natural projection $\pi: \mathcal{O}_{\mu^n} \rightarrow \mathcal{O}_{\mu^1}$.

Given p and $s \in \mathcal{O}_{\mu^n}^T$, let $\Sigma(p, s)$ be the set of paths in the canonical graph (V, E) associated to \mathcal{O}_{μ^n} , and let $\overline{\Sigma}(p, s) \subset \Sigma(p, s)$ be the subset of horizontal paths. Given any $\gamma \in \overline{\Sigma}(p, s)$, by Lemmas 4.10 and 4.11, its projection $\tilde{\gamma} = \pi(\gamma)$ is an ascending path in the GKM graph $(\tilde{V}, \tilde{E}_{\text{GKM}})$ associated to \mathcal{O}_{μ^1} . Note that, in ascending order, the T -fixed points of \mathcal{O}_{μ^1} are

$$-x_1, -x_2, \dots, -x_n, x_n, \dots, x_2, x_1;$$

however, $(\tilde{V}, \tilde{E}_{\text{GKM}})$ is not complete because it doesn't contain the edge $(-x_j, x_j)$ for any j . Finally, given a path $\tilde{\gamma}$ in $(\tilde{V}, \tilde{E}_{\text{GKM}})$, let $V(\tilde{\gamma})$ be the set of vertices of $\tilde{\gamma}$. To state our main theorem, we will need the following definitions.

Definition 5.17. A path $\gamma \in \overline{\Sigma}(p, s)$ with $\pi(\gamma) = \tilde{\gamma}$ is **incomplete** if $\{-\pi(s), \pi(s)\} \subset V(\tilde{\gamma})$. Otherwise γ is **complete**.

Definition 5.18. A path $\gamma \in \overline{\Sigma}(p, s)$ with $\pi(\gamma) = \tilde{\gamma}$ is **relevant** if either it is complete or if it is incomplete and $x_{k(\gamma)+1} \in V(\tilde{\gamma})$, where $k(\gamma) = \max\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\}$. (Observe that if γ is incomplete then by definition $\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\} \neq \emptyset$ and since $(-x_n, x_n) \notin \tilde{E}_{\text{GKM}}$, $k(\gamma) < n$.)

For every p and $s \in \mathcal{O}_{\mu^n}^T$ and every path $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ in $\overline{\Sigma}(p, s)$ define

$$P(\gamma) = \tilde{\Lambda}_{\pi(s)}^- \prod_{i=1}^{|\gamma|} \frac{1}{\pi(s) - \pi(\gamma_i)} \frac{\pi(\gamma_{i+1}) - \pi(\gamma_i)}{\eta(\gamma_i, \gamma_{i+1})}$$

The main theorem of this section can be stated as follows.

Theorem 5.19. Let $\pi: \mathcal{O}_{\mu^n} \rightarrow \mathcal{O}_{\mu^1}$ be the natural projection map. Let $\varphi = \psi_n^\xi: \mathcal{O}_{\mu^n} \rightarrow \mathbb{R}$ be a generic component of the moment map. Consider the canonical classes $\alpha_p \in H_T^*(\mathcal{O}_{\mu^n}; \mathbb{Z})$ for all $p \in \mathcal{O}_{\mu^n}^T$. For all $s \in \widehat{M}_q^T$ let $\hat{\alpha}_s \in H_T^*(\widehat{M}_q; \mathbb{Z})$ be the canonical class on the fiber $\widehat{M}_q = \pi^{-1}(\pi(q))$.

- (1) Given $s \in \widehat{M}_q^T$, let $R(p, s) \subset \overline{\Sigma}(p, s)$ denote the set of relevant paths from p to s . Then

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in R(p, s)} Q(\gamma) \right) \hat{\alpha}_s(q), \quad \text{where for every } \gamma \in R(p, s)$$

$$Q(\gamma) = \begin{cases} P(\gamma) & \text{if } \gamma \text{ is complete, and} \\ P(\gamma) \frac{2\pi(s)}{\pi(s) + x_{k(\gamma)+1}} & \text{if } \gamma \text{ is incomplete.} \end{cases}$$

- (2) $Q(\gamma)$ is the product of distinct positive roots.

Before proving Theorem 5.19 we need to analyze $P(\gamma)$.

Proposition 5.20. Let $\gamma = (\gamma_1, \dots, \gamma_{|\gamma|+1})$ be a path in $\overline{\Sigma}(p, s)$. Let $\tilde{\gamma} = \pi(\gamma)$, and let $SV(\tilde{\gamma})$ be the skipped vertices of $\tilde{\gamma}$. Then

$$P(\gamma) = \begin{cases} \prod_{r \in SV(\tilde{\gamma}) \setminus \{-\pi(s)\}} \eta(r, \pi(s)) & \text{if } \gamma \text{ is complete, and} \\ \frac{1}{2\pi(s)} \prod_{r \in SV(\tilde{\gamma})} \eta(r, \pi(s)) & \text{if } \gamma \text{ is incomplete.} \end{cases}$$

Proof. Let $m(r, r')$ be the magnitude of an edge $(r, r') \in \tilde{E}_{\text{GKM}}$ with respect to $\tilde{\psi}$. Every edge has magnitude 1. \square

We are now ready to prove Theorem 5.19

Proof of Theorem 5.19. Since Proposition 5.6 implies that π is a strong symplectic fibration, and since $\tilde{\psi}_1: \mathcal{O}_{\mu^1}^T \rightarrow \mathfrak{t}^*$ is the inclusion, Theorem 1.10, Corollary 4.7 and Proposition 5.1 together imply that

$$\alpha_p(q) = \sum_{s \in \widehat{M}_q^T} \left(\sum_{\gamma \in \overline{\Sigma}(p, s)} P(\gamma) \right) \widehat{\alpha}_s(q).$$

If $\gamma \in \overline{\Sigma}(p, s)$ is complete, then γ is relevant and $P(\gamma) = Q(\gamma)$. On the other hand, Lemmas 5.14 and 5.15 still hold when $G = SO(2n)$ instead of $SO(2n+1)$. Indeed, the proof is identical, except that in the proof of Lemma 5.14 we no longer need to consider the case that $(-x_l, x_l)$ is an edge of $\tilde{\gamma}$. Hence, as before, the set of incomplete paths can be decomposed into pairs of paths γ and γ' , so that $V(\tilde{\gamma}') = V(\pi(\gamma'))$ is obtained from $V(\tilde{\gamma}) = V(\pi(\gamma))$ by replacing $x_{k(\gamma)+1}$ by $-x_{k(\gamma)+1}$, where $k(\gamma) = \max\{j \mid \{-x_j, x_j\} \subset V(\tilde{\gamma})\}$. Additionally, by the definition of $k(\gamma)$, $\pi(s) \neq \pm x_{k(\gamma)+1}$, and so $\eta(\pm x_{k(\gamma)+1}, \pi(s)) = \pi(s) \mp x_{k(\gamma)+1}$. Hence by Proposition 5.20

$$\begin{aligned} P(\gamma) + P(\gamma') &= \frac{1}{2\pi(s)} \left(\prod_{r \in SV(\tilde{\gamma}) \cap SV(\tilde{\gamma}')} \eta(r, \pi(s)) \right) (\eta(-x_{k(\gamma)+1}, \pi(s)) + \eta(x_{k(\gamma)+1}, \pi(s))) \\ &= \prod_{r \in SV(\tilde{\gamma}) \cap SV(\tilde{\gamma}')} \eta(r, \pi(s)) = Q(\gamma). \end{aligned}$$

As in the previous subsection, this proves part (1) of Theorem 5.19. The proof of part (2) also proceeds analogously to the argument in the previous subsection. \square

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